ENUMERATIVE GEOMETRY OF K3 SURFACES: STABLE PAIR AND BPS INVARIANTS

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Abstract. In this thesis, we study the generating function for the stable pair invariants $e(P_k(X,[C]))$ on a polarised K3 surface (X,H), where the curve class is chosen so that $C \cdot H$ is minimal among positive intersections with the polarisation, following [KY00]. Additionally, we explore the enumeration of partial normalisations of Gorenstein curves and use the obtained results to express the generating function of the stable pair invariants in its BPS form. This allows us to analyse the contributions of individual curves in the linear system $|C|$ to the corresponding BPS numbers. Furthermore, we recover as a limit the famous Yau–Zaslow formula proved by Beauville, and explore the relationship between the BPS invariants for X and the Gromov–Witten theory of the local K3 surface $X \times \mathbb{C}$ under the assumption of the MNOP conjecture.

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Introduction

Curve counting has a long history in algebraic geometry. Already in the 19th century, classical geometers were captivated by the problem of enumerating curves satisfying specified geometric conditions. A celebrated example is the fact that every smooth cubic surface contains exactly 27 lines, a result that has come to symbolise the beauty of the subject.

In the last few decades, the study of curve counting has undergone a profound transformation. Modern approaches have given rise to powerful mathematical frameworks, most prominently Gromov–Witten, Donaldson–Thomas, and Pandharipande–Thomas (or stable pair) invariants, each offering a distinct perspective on the enumerative geometry of curves. These theories are not isolated: deep conjectures, many of which have now been proven, reveal surprising equivalences and correspondences between them, see [PT14] for a summary on the topic. Today, enumerative algebraic geometry is less concerned with computing the raw number of curves satisfying given conditions, and more with uncovering the intricate web of relations between different enumerative invariants.

Let (X, H) be a polarised K3 surface. A pair $(\mathcal{F}, s)$, where $\mathcal{F}$ corresponds to a pure sheaf of dimension 1 on X and s is a section $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ with 0-dimensional cokernel, is called *stable pair*. We denote by $P_k(X, [C])$ the moduli space of stable pairs $(\mathcal{F}, s)$ with curve class $c_1(\mathcal{F}) = [C]$ and Euler characteristic $\chi(\mathcal{F}) = k$. By choosing C such that $C \cdot H$ is minimal among positive intersections with the polarisation, $P_k(X, [C])$ coincides with the moduli space of coherent systems $\mathrm{Syst}^1(0, [C], k)$ constructed by Le Potier, [Le 93]. Hence, $P_k(X, [C])$ is a projective scheme.

In this thesis, we study the generating function for the stable pair invariants $e(P_k(X, [C]))$ on a polarised K3 surface (X, H) introduced in [PT07], where the curve class is chosen so that $C \cdot H$ is minimal among positive intersections with the polarisation, following [KY00]. Additionally, we explore the enumeration of partial normalisations of Gorenstein curves and use the obtained results to express the generating function of the stable pair invariants in its BPS form. This allows us to analyse the contributions of individual curves in the linear system $|C|$ to the corresponding BPS numbers. Furthermore, we recover as a limit the famous Yau–Zaslow formula proved by Beauville, and explore the relationship between the BPS invariants for X and the Gromov–Witten theory of the local K3 surface $X \times \mathbb{C}$ under the assumption of the MNOP conjecture.

In the first section, we study Beauville's proof of the Yau–Zaslow formula

$$\sum_{G \geqslant 0} e(G) q^G = \prod_{n \geqslant 1} (1 - q^n)^{-24},$$

where $e(G)$ denotes the number (up to multiplicity) of rational curves in a linear system of integral curves of arithmetic genus G and of dimension G on a K3 surface, [Bea97]. Additionally, we explore the relation between curve singularities and the multiplicity of the rational curves.

In the second section, we prove that given an integral Gorenstein curve C of arithmetic genus G and geometric genus $\tilde{\mathrm{G}}$, we have the following relation

$$\sum_i e(C^{[i]})q^{i+1-G} = \sum_{\tilde{\mathrm{G}}\leqslant g\leqslant G} n_g(C)F_g(q),$$

where $e(C^{[i]})$ denotes the topological Euler characteristic of the Hilbert scheme of points $C^{[i]}$, and $F_g(q)$ is defined as follows. For a smooth curve Σ of genus g, we set

$$F_g(q) := \sum_{i\geqslant 0} e(\Sigma^{(i)})q^{i+1-g},$$

where $e(\Sigma^{(i)})$ denotes the topological Euler characteristic of the i-th symmetric product of Σ. Additionally, we show that $n_g(C)$ are integers and, if C is a nodal curve, $n_g(C)$ counts precisely the partial normalisations of C of arithmetic genus g.

In the third section, under the assumption that $[C]$ is a curve class on a polarised K3 surface (X, H) such that $C \cdot H$ is minimal among positive intersections with the polarisation and $C^2 = 2G-2$, we prove the Yau–Zaslow's type generating function for the total moduli of stable pairs

$$\sum_{G\geqslant 0}\sum_{d\geqslant 0} e(P_{d+1-G}(X,[C_G]))q^{G-1}y^{d+1-G} = \frac{(y^{-1/2}-y^{1/2})^{-2}}{q\prod_{n\geqslant 1}(1-q^n)^{20}(1-q^n y)^2(1-q^n y^{-1})^2}.$$

The proof of the above formula is based on [KY00], however we incorporate modified versions of some proofs of Yoshioka, see Lemmas 3.3 and 3.4. Furthermore, we use our calculations from Section 2 to prove the BPS form

$$\sum_{G\geqslant 0}\sum_{k\geqslant 1-G} e(P_k(X,[C_G]))y^k q^G = \sum_{G\geqslant 0}\sum_{g\geqslant 0} N_{g,G}y^{1-g}(1-y)^{2g-2}q^G,$$

where $N_{g,G}$ are integers. This relation allow us to recover the usual Yau–Zaslow formula in the limit $y \longrightarrow 1$, giving us an interpretation of the BPS invariants $N_{0,G}$. Additionally, we explore the contributions of curves in the linear system $|C_G|$ to the BPS invariants $N_{g,G}$ and relate them to the integers $n_g(C)$ introduced in Section 2. This allows us to conclude $N_{g,G} = 0$ for $g > G$.

Finally, in the fourth section we explore, under the assumption of the MNOP conjecture, the relation between the BPS invariants $N_{g,G}$ from Section 3 and the Gromov-Witten invariants of the local K3 surface $X \times \mathbb{C}$.

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1. The Yau–Zaslow formula

Let X be a K3 surface with a linear system $|C_G|$ of integral curves of arithmetic genus G and of dimension G. Yau and Zaslow conjectured the following generating function, [YZ96]:

$$\sum_{G\geqslant 0} e(G)q^G = \prod_{n\geqslant 1}(1-q^n)^{-24}, \tag{1.1}$$

where $e(G)$ denotes the number of rational curves in the linear system $|C_G|$. In this section, we study Beauville's proof of this generating function, [Bea97].

Firstly, we relate the coefficients $e(G)$ with the topological Euler characteristic of the Hilbert scheme of G-points $e(X^{[G]})$ via the Göttsche's formula.

Theorem 1.1 ([Göt90])**.** *Let X be a smooth projective surface over $\mathbb{C}$ or $\mathbb{F}_p$. Then, we have*

$$\sum_{G\geqslant 0} e(X^{[G]})q^G = \prod_{n\geqslant 1}(1-q^n)^{-e(X)}.$$

By the Göttsche's formula, for a K3 surface we have $e(G) = e(X^{[G]})$. We aim to show that $e(X^{[G]})$ counts (up to multiplicity) rational curves in any G-dimensional linear system $|C_G|$ of integral curves on X of arithmetic genus G. In order to do this, we relate $X^{[G]}$ to the compactified Jacobian of the family of curves $\mathcal{C}_G \longrightarrow |C_G|$ associated to the linear system $|C_G|$, which is defined as follows.

Definition 1.2. Let $X \longrightarrow S$ be a flat, finitely presented, locally projective morphism of schemes, whose geometric fibres are integral curves. Then, we define the moduli functor

$$\begin{aligned} \overline{\operatorname{Pic}}_{X/S}\colon (\operatorname{Sch}/S)^{op} &\longrightarrow \text{Sets}, \\ T &\longmapsto \{\mathcal{F} \in Mod_{\mathcal{O}_{X_T}} : \mathcal{F}\ T\text{-flat},\ \mathcal{F}_t \text{ torsion free of rank 1 for } t \in T\}/\sim, \end{aligned}$$

where $\mathcal{F} \sim \mathcal{G}$ if there exist a line bundle $\mathcal{L} \in \operatorname{Pic}(T)$ and an isomorphism $\mathcal{F} \otimes q^*\mathcal{L} \simeq \mathcal{G}$ for $q\colon X \times T \longrightarrow T$ the canonical projection. Furthermore, after fixing a very ample line bundle $\mathcal{O}_X(1)$, we set $\overline{\operatorname{Pic}}^n_{X/S}$ to be the open sub-functor of relative torsion free sheaves of rank 1 with Hilbert polynomial n.
The étale sheafifications of the above moduli functors, $\overline{\operatorname{Pic}}_{X/S,\text{ét}}$ and $\overline{\operatorname{Pic}}^n_{X/S,\text{ét}}$, are representable by S-schemes, see Theorem 8.1. in [AK80]. We call $\overline{\operatorname{Pic}}^0_{X/S,\text{ét}}$ the *compactified Jacobian* of the family $X \longrightarrow S$, and we drop the subindex "ét" in this document.

Theorem 1.3. *Let $\mathcal{C}_G \longrightarrow |C_G|$ be a G-dimensional linear system of integral curves on a K3 surface X of arithmetic genus G. Then, its associated compactified Jacobian $\overline{\mathrm{Pic}^0_{\mathcal{C}_G}}$ is birationally equivalent in codimension 1 to the Hilbert scheme of points $X^{[G]}$.*

Proof. Let $U \subset \overline{\mathrm{Pic}^G_{\mathcal{C}_G}}$ be the open subscheme consisting of pairs $(C_t, \mathcal{L})$, where $\mathcal{L}$ is a line bundle on C_t with $h^0(C_t, \mathcal{L}) = 1$. This condition ensures that we can assign to each pair $(C_t, \mathcal{L})$ a unique effective Cartier divisor of degree G, say $D(\mathcal{C}_t, \mathcal{L})$. On the other hand, let $V \subset X^{[G]}$ be the subscheme of divisors contained in exactly one fibre of $\mathcal{C}_G \longrightarrow |C|$. We claim that the morphism

$$U \subset \overline{\mathrm{Pic}^G_{\mathcal{C}_G}} \longrightarrow V \subset X^{[G]},\ (C_t, \mathcal{L}) \longmapsto D(C_t, \mathcal{L})$$

is well-defined and is an isomorphism.
To verify that $D(C_t, \mathcal{L})$ is not contained in the intersection of two elements of the linear system $|C|$, it is enough to show that $\mathcal{L}^\vee \otimes \mathcal{O}_{C_t}(C_t)$ does not correspond to an effective divisor. Given $(C_t, \mathcal{L}) \in U$, we have $\omega_{C_t} = \mathcal{O}_{C_t}(C_t)$ via the adjunction formula. By Serre duality and Riemann-Roch, we have $h^0(C_t, \mathcal{L}^\vee \otimes \mathcal{O}_{C_t}(C_t)) = h^1(C_t, \mathcal{L}) = 0$. Hence, $\mathcal{L}^\vee \otimes \mathcal{O}_{C_t}(C_t)$ is not effective. Hence, the above morphism is well-defined and it is clearly an isomorphism.
We have that $\mathrm{codim}(\overline{\mathrm{Pic}^G_{\mathcal{C}_G}} - U) \geqslant 2$ because generic effective divisors on a curve $C \in |C_G|$ are not contained in the intersection of two or more curves of the linear system $|C_G|$. Similarly, $\mathrm{codim}(X^{[G]} - V) \geqslant 2$.
Finally, by tensoring with a line bundle of degree G, we get an isomorphism $\overline{\mathrm{Pic}^0_{\mathcal{C}_G}} \simeq \overline{\mathrm{Pic}^G_{\mathcal{C}_G}}$. □

The compactified Jacobian $\overline{\mathrm{Pic}^0_{\mathcal{C}_G}}$ is an open subscheme of the moduli of simple sheaves on X, Spl_X. Then, the symplectic structure constructed in Appendix 1 induces a symplectic structure on $\overline{\mathrm{Pic}^0_{\mathcal{C}_G}}$. Moreover, $X^{[G]}$ is irreducible symplectic, see [Muk84]. Since $\overline{\mathrm{Pic}^0_{\mathcal{C}_G}}$ and $X^{[G]}$ are birational equivalent in codimension 1 and irreducible symplectic, they are deformation equivalent, see [Huy96]. In particular, $e(\overline{\mathrm{Pic}^0_{\mathcal{C}_G}}) = e(X^G)$. Hence, we have

$$\sum_{G \geqslant 0} e(\overline{\mathrm{Pic}^0_{\mathcal{C}_G}}) q^G = \prod_{n \geqslant 1} (1 - q^n)^{-24}.$$

Now, we study the contributions of each curve in the family $\mathcal{C}_G$ to $e(\overline{\mathrm{Pic}^0_{\mathcal{C}_G}})$.

Lemma 1.4. *Let C be a curve and let $\overline{\mathrm{Pic}^0}(C)$ be its compactified Jacobian. For $\mathcal{L} \in \overline{\mathrm{Pic}^0}(C)$, we consider its associated partial normalisation $\pi \colon C' = \mathrm{Spec}(\mathcal{E}nd(\mathcal{L})) \longrightarrow C$. Then, there exists an $\mathcal{O}_{C'}$-module of rank 1, $\mathcal{L}'$, such that $\mathcal{L} \simeq \pi_* \mathcal{L}'$.*

Proof. We have that $\mathcal{E}nd(\mathcal{L})$ is an $\mathcal{O}_C$-subalgebra of the sheaf of rational functions on C. Additionally, via Cayley–Hamilton Theorem, $\mathcal{E}nd(\mathcal{L})$ is a finitely generated $\mathcal{O}_C$-module because $\mathcal{L}$ is coherent. Hence, $\mathcal{E}nd(\mathcal{L})$ is contained in $\mathcal{O}_{\tilde{C}}$ for $\tilde{C}$ the normalisation of C. Given an $\mathcal{O}_C$-subalgebra that is contained in $\mathcal{O}_{\tilde{C}}$, we can define a partial normalisation of C. Let $\pi' \colon C' \longrightarrow C$ be the partial normalisation corresponding to $\mathcal{E}nd(\mathcal{L})$, then $\pi'_* \mathcal{O}_{C'} \simeq \mathcal{E}nd(\mathcal{L})$. Via the above identification $\mathcal{L}$ is a $\pi_* \mathcal{O}_{C'}$-module, so $\mathcal{L}$ corresponds to $\pi'_* \mathcal{L}'$, where $\mathcal{L}'$ is some $\mathcal{O}_{C'}$-module of rank 1. □

Lemma 1.5. *Let $L \in \mathrm{Pic}^0(C)$ and let $\mathcal{L} \in \overline{\mathrm{Pic}^0}(C)$. Denote by $\pi\colon C' \longrightarrow C$ the partial normalisation of C associated to $\mathcal{E}nd(\mathcal{L})$. Then, $\mathcal{L} \otimes L$ is isomorphic to $\mathcal{L}$ if, and only if π^*L is trivial.*

Proof. There is a $\mathcal{L}'$ be a rank 1 $\mathcal{O}_{C'}$-module such that $\mathcal{L} = \pi_*\mathcal{L}'$. Via the projection formula, we have $\mathcal{L} \otimes L \simeq \pi_*(\mathcal{L}' \otimes \pi^*L)$. Hence, if π^*L is trivial, we have $\mathcal{L} \otimes L \simeq \mathcal{L}$. On the other hand, assume that $\mathcal{L} \otimes L \simeq \mathcal{L}$. We have

$$\mathcal{H}om_{\mathcal{O}_C}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{H}om_{\mathcal{O}_C}(\mathcal{L}, \mathcal{L} \otimes L) \simeq \mathcal{E}nd_{\mathcal{O}_C}(\mathcal{L}) \otimes_{\mathcal{O}_C} L \simeq \pi_*\mathcal{O}_{C'} \otimes L \simeq \pi_*\pi^*L,$$

which yields $\mathrm{Hom}(\mathcal{L}, \mathcal{L}) \simeq H^0(C', \pi^*L)$. Let $s \in H^0(C', \pi^*L) \backslash \{0\}$ correspond to $\mathrm{id}_{\mathcal{L}}$. As π^*L is a line bundle of degree 0, $\mathrm{div}(s)$ does not have poles or zeros. Thus $1/s \in H^0(C', (\pi^*L)^\vee) \backslash \{0\}$, and so π^*L is trivial. □

Let us use this Lemma to prove that non-rational curves in the family $\mathcal{C}_G \longrightarrow |C_G|$ do not contribute to $e(\overline{\mathrm{Pic}^0_{\mathcal{C}_G}})$.

Theorem 1.6. *Let C be a proper (reduced) curve over $\mathbb{C}$ and let $\nu\colon \tilde{C} \longrightarrow C$ be its normalisation. Then, we have the short exact sequence*

$$0 \longrightarrow \ker(\nu^*) \longrightarrow \mathrm{Pic}^0(C) \xrightarrow{\nu^*} \mathrm{Pic}^0(\tilde{C}) \longrightarrow 0,$$

where ν^* is the pullback. Furthermore, $\ker(\nu^*)$ is affine, which implies via the structure theorem of commutative affine abelian groups over $\mathbb{C}$ that $\ker(\nu^*) \simeq \mathbb{G}_m^{\oplus d_1} \oplus \mathbb{G}_a^{\oplus d_2}$.

Proposition 1.7. *Let C be an integral curve whose normalisation $\tilde{C}$ has genus $G \geqslant 1$. Then, $e(\overline{\mathrm{Pic}^0}(C)) = 0$.*

Proof. It is enough to show that for any $n > 0$, there exists a group of order n acting freely on $\mathrm{Pic}^0(C)$. Via Theorem 1.6, we have the exact sequence

$$0 \longrightarrow G \longrightarrow \mathrm{Pic}^0(C) \xrightarrow{\pi^*} \mathrm{Pic}^0(\tilde{C}) \longrightarrow 0,$$

where $\tilde{C}$ denotes the normalisation of C and G is a product of additive and multiplicative groups. In particular, G is an injective object in the category of abelian groups and the above sequence splits as a sequence of abelian groups. Denote by s the section of π^*.
As $\tilde{C}$ is smooth of genus G, $\mathrm{Pic}^0(\tilde{C}) \simeq \mathbb{C}^G/\Lambda$ and its n-th torsion subgroup is of the form $(\mathbb{Z}/n)^{2G}$. Hence, via the splitting, we find a subgroup of order n of $\mathrm{Pic}^0(C)$ for any $n > 0$, say $\langle \mathcal{G} \rangle$ for $\mathcal{G} \in \mathrm{Pic}^0(\tilde{C}) \subset \mathrm{Pic}^0(C)$. We consider the action of $\langle \mathcal{G} \rangle$ on $\overline{\mathrm{Pic}^0}(C)$ induced by the tensor product. This action is free. Indeed, let $\mathcal{L} \in \overline{\mathrm{Pic}}^0(C)$, such that $\mathcal{G}^m \otimes \mathcal{L} \simeq \mathcal{L}$, and let $C' = \mathrm{Spec}(\mathcal{E}nd(\mathcal{L}))$ be the partial normalisation of C associated to $\mathcal{L}$, which fits in $\pi\colon \tilde{C} \xrightarrow{\underline{\pi}'} C' \xrightarrow{\pi'} C$. Since $\mathcal{G}^m \otimes \mathcal{L} = \mathcal{L}$, we have that $\pi'^*\mathcal{G}^m \simeq \mathcal{O}_{C'}$ by Lemma 1.5. Hence, $\pi^*\mathcal{G}^m = \underline{\pi}'^*\pi'^*\mathcal{G}^m \simeq \mathcal{O}_{\tilde{C}}$. Then, applying the section s we obtain $\mathcal{G}^m = \mathcal{O}_C$, so the action is free. This implies that for all $n > 0$, n divides $e(\overline{\mathrm{Pic}^0}(C))$, and so $e(\overline{\mathrm{Pic}^0}(C)) = 0$. □

Corollary 1.8. *Denote by $|C_G|_{rat} \subset |C_G|$ the subset of rational curves. Then, $|C_G|_{rat}$ is finite and we have*

$$e(\overline{\mathrm{Pic}^0_{\mathcal{C}_G}}) = \sum_{t \in |C|_{rat}} e(\overline{\mathrm{Pic}^0}(C_t)).$$

Proof. Assume that $|C_G|_{rat}$ is not finite. Then, it contains a curve, which produces a ruling of the K3 surface X. This is a contradiction. Furthermore, note that given a surjective morphism $f\colon Y \longrightarrow Z$ of complex projective varieties such that the topological Euler characteristic of its fibres is trivial, we have $e(Y) = 0$. Indeed, this is clear if f is a locally trivial fibration. In the general case, there exists an stratification of f such that f is a locally trivial fibration on each stratum, [Ver76]. Consider the morphism $p\colon \overline{\mathrm{Pic}^0_{\mathcal{C}_G}} \longrightarrow |C_G|$ restricted to $p^{-1}(|C_G| - |C_G|_{rat})$. Then, by Proposition 1.7, we have $e(p^{-1}(|C_G| - |C_G|_{rat})) = 0$. □

The previous Corollary yields

$$e(G) = \sum_{t \in |C|_{rat}} e(\overline{\mathrm{Pic}^0}(C_t)),$$

where $|C_G|_{rat} \subset |C_G|$ denotes the rational locus. Hence, we interpret $e(\overline{\mathrm{Pic}^0}(C_t))$ as the multiplicity of the curve C_t. We show now that, if C_t is a nodal curve, $e(\overline{\mathrm{Pic}^0}(C_t)) = 1$, and explore the relation between the singularities of C_t and the value of $e(\overline{\mathrm{Pic}^0}(C_t))$.

Proposition 1.9. *Let C be an integral rational curve. Denote by $\hat{C} \longrightarrow C$ its minimal unibranch partial normalisation. Then, we have $e(\overline{\mathrm{Pic}^0}(C)) = e(\overline{\mathrm{Pic}^0}(\hat{C}))$.*

Proof. By Proposition 1.7, the claim holds for non-rational curves. Assume that C is a rational curve. Denote its singular locus by Σ, and its preimage along the normalisation $\pi\colon \tilde{C} \longrightarrow C$ by $\tilde{\Sigma}$. To show $e(\overline{\mathrm{Pic}^0}(C)) = e(\overline{\mathrm{Pic}^0}(\hat{C}))$, it is enough to prove that for any $n \geqslant |\tilde{\Sigma}|$, there exists a line bundle $\mathcal{L}_n \in \mathrm{Pic}^0(C)$ of order n such that $\mathcal{L}_n$ acts freely on $\overline{\mathrm{Pic}^0}(C) - \pi_*(\overline{\mathrm{Pic}^0}(\hat{C}))$.
We have the exact sequence

$$1 \longrightarrow \mathcal{O}^*_C \longrightarrow \mathcal{O}^*_{\tilde{C}} \longrightarrow \mathcal{O}^*_{\tilde{C}}/\mathcal{O}^*_C \longrightarrow 1,$$

from which follows the isomorphism $H^0(C, \mathcal{O}^*_{\tilde{C}}/\mathcal{O}^*_C) \xrightarrow{\simeq} \mathrm{Pic}^0(C)$. Then, we aim to construct corresponding global sections of $\mathcal{O}^*_{\tilde{C}}/\mathcal{O}^*_C$. The evaluations $\mathcal{O}^*_C \longrightarrow \oplus_\Sigma \mathbb{C}^*$ and $\mathcal{O}^*_{\tilde{C}} \longrightarrow \oplus_{\tilde{\Sigma}} \mathbb{C}^*$ produce a surjective homomorphism $\mathcal{O}^*_{\tilde{C}}/\mathcal{O}^*_C \longrightarrow \oplus_{\tilde{\Sigma}} \mathbb{C}^* / \oplus_\Sigma \mathbb{C}^*$. For any integer $n \geqslant |\tilde{\Sigma}|$ we can find a section s in a neighbourhood of $\tilde{\Sigma}$ such that the values $s(x_i)$ are different for each $x_i \in \tilde{\Sigma}$, and such that n is minimal with s^n being a section of $\mathcal{O}_C$. Let $\mathcal{L}_n$ be the line bundle associated to s.
Given $\mathcal{F} \in \overline{\mathrm{Pic}^0}(C) - \pi_*(\overline{\mathrm{Pic}^0}(\hat{C}))$, let $\pi'\colon C' = \mathrm{Spec}(\mathcal{E}nd(\mathcal{F})) \longrightarrow C$ be the partial normalisation associated to $\mathcal{F}$, such that $\mathcal{E}nd(\mathcal{F}) \simeq \pi'_* \mathcal{O}_{C'}$. Note that C' is not unibranch, otherwise we have a factorisation $C' \longrightarrow \hat{C} \longrightarrow C$, which contradicts that $\mathcal{F} \notin \pi_*(\overline{\mathrm{Pic}^0}(\hat{C}))$. Thus, there are two points in $\tilde{\Sigma}$ mapping to the same point in C', which implies that the section s associated to $\mathcal{L}_n$ does not belong to $\mathcal{O}^*_{C'}$ since s takes distinct values on different elements of $\tilde{\Sigma}$ by construction.

As $\mathcal{O}_C \hookrightarrow \mathcal{O}_{C'}$, we have a commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}^*_{\tilde{C}}/\mathcal{O}^*_C) & \xrightarrow{\simeq} & \mathrm{J}^0(C) \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}^*_{\tilde{C}}/\mathcal{O}^*_{C'}) & \xrightarrow{\simeq} & \mathrm{J}^0(C') \end{array},$$

from which we conclude that the pullback of $\mathcal{L}_n$ to $\mathrm{Pic}^0(C')$ is non-trivial. By Lemma 1.5, we have that $\mathcal{L}_n \otimes \mathcal{F}$ is not isomorphic to $\mathcal{F}$. □

Corollary 1.10. *For a rational nodal curve C, we have $e(\overline{\mathrm{Pic}^0}(C)) = 1$.*

Proof. The minimal unibranch normalisation of a rational nodal curve is $\mathbb{P}^1$. Hence,

$$e(\overline{\mathrm{Pic}^0}(C)) = e(\overline{\mathrm{Pic}^0}(\mathbb{P}^1)) = e(\{*\}) = 1.$$

□

Finally, in Appendix 2 we explore how $e(\overline{\mathrm{Pic}^0}(C))$ depends on the singularities of C for the case of simple singularities.

2. Counting partial normalisations

In this section, we prove that given an integral Gorenstein curve C of arithmetic genus G and geometric genus $\tilde{\mathrm{G}}$, we have the following relation

$$\sum_i e(C^{[i]})q^{i+1-G} = \sum_{\tilde{\mathrm{G}} \leqslant g \leqslant G} n_g(C)F_g(q),$$

where $e(C^{[i]})$ denotes the topological Euler characteristic of the Hilbert scheme of points $C^{[i]}$, and $F_g(q)$ is defined as follows. For a smooth curve X_g of genus g, we set

$$F_g(q) := \sum_{i \geqslant 0} e(X_g^{(i)})q^{i+1-g},$$

where $e(X_g^{(i)})$ denotes the topological Euler characteristic of the i-th symmetric product of X_g. Additionally, we show that $n_g(C)$ are integers and, if C is a nodal curve, $n_g(C)$ counts precisely the partial normalisations of C of arithmetic genus g. This section grew out of hints and questions suggested by Prof. Richard Thomas, for which I am very grateful.

Lemma 2.1. *Let X_g be a smooth curve of genus g. Then, for $|q| < 1$, we obtain*

$$F_g(q) := \sum_{i \geqslant 0} e(X_g^{(i)})q^{i+1-g} = q^{1-g}(1-q)^{2g-2}.$$

Proof. We show that for any topological space X we have $\sum_{i \geqslant 0} e(X^{(i)})q^i = (1-q)^{-e(X)}$. Note that $e(X^{(i)})$ only depends on $e(X)$. By the identity $(X \sqcup \{*\})^{(i)} = X^{(i)} \sqcup (X \sqcup \{*\})^{(i-1)}$, we have

$$\sum_{i \geqslant 0} e(X^{(i)})q^i = (1-q)\sum_i e((X \sqcup \{*\})^{(i)})q^i.$$

Since $e(X \sqcup \{*\}) = e(X) + 1$, we may assume $e(X) \geqslant 0$; otherwise, we add sufficiently many points to X. Moreover, as our claim only depends on $e(X)$, we may reduce to the case where

X consists of $e(X)$ points. In this situation, $e(X^{(i)})$ counts unordered length i tuples of points in X, and so

$$e(X^{(i)}) = \binom{i+e(X)-1}{i}.$$

Substituting these values gives the desired result. □

Let C be an integral Gorenstein curve of arithmetic genus G and let $\mathcal{M}(1, i+1-G) = \overline{\mathrm{Pic}}^i(C)$ be the moduli space of rank 1 stable sheaves on C and of Euler characteristic $i+1-G$. One can see that $\mathcal{M}(1, i+1-G)$ is a compactification of $\mathrm{Pic}^i(C)$ and such stable sheaves correspond exactly to the torsion free sheaves of rank 1 on C with corresponding Euler characteristic, and so we identify it with the compactified Picard scheme $\overline{Pic}^i(C)$. Consider the morphism

$$\varphi_i \colon C^{[i]} \longrightarrow \mathcal{M}(1, i+1-G) = \overline{\mathrm{Pic}}^i(C),\ D \longmapsto \mathcal{I}_D^\vee,$$

where $\mathcal{I}_D$ is the ideal sheaf associated to the generalised divisor D. For a general divisor D in $C^{[i]}$, whose support intersects the singular locus of C, its associated ideal sheaf $\mathcal{I}_D$ is not a line bundle. However, it is a torsion free sheaf of rank 1. Hence, φ_i is well-defined. The fibres of the above morphism are given by $\varphi_i^{-1}(\mathcal{F}) = \mathbb{P}(H^0(\mathcal{F}))$, see [Har86].
Additionally, consider the morphism

$$\psi_i \colon \mathcal{M}(1, i+1-G) \longrightarrow \mathcal{M}(1, -(i+1-G)),\ \mathcal{F} \longmapsto \mathcal{H}om_{\mathcal{O}_C}(\mathcal{F}, \omega_C) = \mathcal{F}^\vee \otimes \omega_C, \tag{2.1}$$

which is an isomorphism. Indeed, define $\phi \colon \mathcal{M}(1, -(i+1-G)) \longrightarrow \mathcal{M}(1, i+1-G)$ given by $\mathcal{G} \longmapsto \mathcal{G}^\vee \otimes \omega_C$. Note that $\phi\psi_i(\mathcal{F}) = \mathcal{F}^{\vee\vee}$ and $\psi_i\phi(\mathcal{G}) = \mathcal{G}^{\vee\vee}$. Hence, we conclude via the following lemma.

Lemma 2.2. *Let C be an integral Gorenstein curve. Then, any torsion free coherent sheaf on C is reflexive, i.e. the natural morphism $\mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$ is an isomorphism.*

Proof. See Lemma 1.1. in [Har86]. □

Via Serre duality, we have that

$$H^1(\mathcal{F})^\vee = Ext^1(\mathcal{O}_C, \mathcal{F})^\vee = Hom(\mathcal{F}, \omega_C) = H^0(C, \mathcal{H}om(\mathcal{F}, \omega_C)),$$

where the last equality follows by the global-local Ext spectral sequence as $\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_C) = 0$ for any torsion free sheaf $\mathcal{G}$ of rank 1 on a Gorenstein curve, see Lemma 2.3.

Lemma 2.3. *Let C be an integral Gorenstein curve and let $\mathcal{G}$ be a torsion free coherent sheaf on C. Then, $\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_C) = 0$ for $i > 0$.*

Proof. See Lemma 1.1. in [Har86]. □

Consider following (locally closed) stratification via $h^0(\mathcal{F})$:

$$\mathcal{M}(1, i+1-G) = \bigsqcup_{k \geqslant 0} S_k, \text{ where } S_k := \{\mathcal{F} \in \mathcal{M}(1, i+1-G) : h^0(\mathcal{F}) = k\}. \tag{2.2}$$

For $\mathcal{F} \in \mathcal{M}(1, i+1-G)$ we have $i+1-G = \chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F})$, so the above stratification can equivalently be described by fixing $h^1(\mathcal{F})$. In particular, the stratification (2.2) induces an stratification on $\mathcal{M}(1, -(i+1-G))$ as $h^0(\mathcal{H}om_{\mathcal{O}_C}(\mathcal{F}, \omega_C)) = h^1(\mathcal{F})$, which is compatible with the isomorphism defined in (2.1). This observation is the key step toward the following result.

Lemma 2.4. *Let C be an integral Gorenstein curve of arithmetic genus G. Then, we have*

$$e(C^{[i+G-1]}) - e(C^{[-i+G-1]}) = i \cdot e(\overline{\mathrm{Pic}^0}(C)),$$

where $\overline{\mathrm{Pic}^0}(C) \simeq \mathcal{M}(1, i+1-G)$ denotes the compactified Jacobian of C.

Proof. We show the following equivalent relation

$$e(C^{[i]}) - e(C^{[2G-2-i]}) = (i+1-G)e(\overline{\mathrm{Pic}^0}(C)).$$

Let $\psi_i^{-1} \circ \varphi_{2G-2-i} \colon C^{[2G-2-i]} \longrightarrow \mathcal{M}(1, i+1-G)$, where ψ_i and φ_{2G-2-i} are the morphisms defined above. This morphism has fibres

$$(\psi_i^{-1} \circ \varphi_{2G-2-i})^{-1}(\mathcal{F}) = \mathbb{P}(H^0(C, \mathcal{H}om_{\mathcal{O}_C}(\mathcal{F}, \omega_C))).$$

Hence, the stratification (2.2) yields

$$e(C^{[2G-2-i]}) = \sum_{k \geqslant 0} h^0(\mathcal{H}om(\mathcal{F}, \omega_C), \mathcal{F} \in S_k) e(S_k) = \sum_{k \geqslant 0} (k - (i+1-G)) e(S_k).$$

Similarly, the morphism $\varphi_i \colon C^{[i]} \longrightarrow \mathcal{M}(1, i+1-G)$ and the stratification (2.2) produce

$$e(C^{[i]}) = \sum_{k \geqslant 0} k e(S_k).$$

Finally, we have an isomorphism

$$\mathcal{M}(1, i+1-G) = \overline{\mathrm{Pic}^i}(C) \overset{\otimes \mathcal{L}}{\simeq} \overline{\mathrm{Pic}^0}(C)$$

induced by tensoring with any line bundle of degree i. This isomorphism and the previous two equations yield the desired result. □

Lemma 2.5. *Let $F(q) = \sum_\chi a_\chi q^\chi$ be a Laurent series with $a_\chi - a_{-\chi} = \chi c$, where c is a constant. Then, there exist coefficients n_g such that*

$$F(q) = \sum_{g \geqslant 0} n_g q^{1-g} (1-q)^{2g-2}.$$

Furthermore, the coefficients n_g are integers if, and only if the coefficients a_χ are integers.

Proof. For $|q| < 1$, we have

$$\begin{aligned} F(q) &= \sum_{\chi > 0} a_{-\chi} q^{-\chi} + a_0 + \sum_{\chi > 0} a_\chi q^\chi = \sum_{\chi > 0} a_\chi (q^\chi + q^{-\chi}) + c \sum_{\chi > 0} (-\chi) q^{-\chi} + a_0 \\ &= \sum_{\chi > 0} a_\chi (q^\chi + q^{-\chi}) + cq(1-q)^{-2} + a_0. \end{aligned}$$

By induction we see that any Laurent polynomial $q^i + q^{-i}$ can be written as linear combination of the rational functions $\{F_g\}_{1 \leqslant g \leqslant i+1}$ with coefficients in $\mathbb{Z}$, where $F_g(q) := q^{1-g}(1-q)^{2g-2}$. Furthermore, both $\{q^i + q^{-i}\}_i$ and $\{F_g(q)\}_g$ are bases of the vector space of Laurent polynomials invariant under the transformation $q \longmapsto q^{-1}$. This ensures that n_g are integers if, and only if a_χ are integers. □

Theorem 2.6. *Let C be an integral Gorenstein curve of arithmetic genus G. Then, there exist integers $n_g(C) \in \mathbb{Z}$, such that*

$$\sum_{i \geqslant 1-G} e(C^{[i+G-1]})q^i = \sum_{0 \leqslant g \leqslant G} n_g(C)F_g(q), \tag{2.3}$$

where $F_g(q) := \sum_{i \geqslant 0} e(\Sigma^{[i]})q^{i+1-g} = q^{1-g}(1-q)^{2g-2}$ for a smooth curve Σ of genus g.

Proof. By Lemma 2.4, $\sum_{i=1-G}^{\infty} e(C^{[i+G-1]})q^i$ satisfies the condition of Lemma 2.5. Thus, we have

$$\sum_{i \geqslant 1-G} e(C^{[i+G-1]})q^i = \sum_{g \geqslant 0} n_g(C)F_g(q).$$

Note that $\sum_{0 \leqslant g \leqslant N} n_g F_g(q)$ has a pole of order $N-1$ at 0. Since the left hand side does not have poles of order greater than $G-1$ at 0, we conclude that the sum on the right hand side runs over $0 \leqslant g \leqslant G$. □

In the previous Theorem, we established that $n_g(C) = 0$ for all $g > G$. Since we aim to interpret the integers $n_g(C)$ as counting (up to multiplicity) partial normalisations of C of arithmetic genus g, we also expect $n_g(C) = 0$ for all $g < \tilde{\mathrm{G}}$, where $\tilde{\mathrm{G}}$ denotes the the geometric genus of C. This is indeed the case for integral Gorenstein curves.

Theorem 2.7. *Let C be an integral Gorenstein curve of arithmetic genus $p_a(C) = G$ and geometric genus $p_g(C) = \tilde{\mathrm{G}}$. Then, we have*

$$\sum_{i \geqslant 1-G} e(C^{[i]})q^i = \sum_{\tilde{\mathrm{G}} \leqslant g \leqslant G} n_g(C)F_g(q).$$

Proof. The central idea is to compare the invariants $n_g(C)$ with the corresponding invariants $n_g(C_0)$, where C_0 denotes a rational curve with the same singularities of C.

Denote by C_{sm} and C_{sg} the smooth and singular locus of C, respectively. We have the stratification

$$C^{[i]} = \bigsqcup_{0 \leqslant l \leqslant i} C_{sg}^{[i-l]} \times C_{sm}^{[l]}, \tag{2.4}$$

where we write a divisor $D \in C^{[i]}$ as $D = D_{sm} \sqcup D_{sg}$ for D_{sm} and D_{sg} the base changes of D along the immersions $C_{sm} \hookrightarrow C$ and $C_{sg} \hookrightarrow C$, respectively.
By smoothness, we have $C_{sm}^{[l]} = C_{sm}^{(l)}$. Then,

$$\sum_{i \geqslant 0} e(C^{[i]})q^i = \sum_{i \geqslant 0} [\sum_{0 \leqslant l \leqslant i} e(C_{sg}^{[i-l]})e(C_{sm}^{(l)})]q^i = \sum_{i \geqslant 0} e(C_{sg}^{[i]})q^i(1-q)^{-e(C_{sm})},$$

where we used $\sum_{j \geqslant 0} e(C_{sm}^{(j)})q^j = \sum_{j \geqslant 0} e(X_G^{(j)})q^j = (1-q)^{-e(C_{sm})}$ for X_G any smooth curve of arithmetic genus G, see Lemma 2.1. A direct calculation shows that $e(C_{sm}) = e(X_G)$, and so $e(C_{sm}^{(j)}) = e(X_G^{(j)})$ for any j.
Let C_0 be a rational curve with the same singularities as C, such that we obtain C from C_0 after attaching $\tilde{\mathrm{G}}$ handles away from its singular locus. By construction, $e(C_{sg}^{[i]}) = e(C_{0,sg}^{[i]})$. Furthermore, C_{sm} is obtained from $C_{0,sm}$ after attaching $\tilde{\mathrm{G}}$ handles. A Mayer–Vietories computation

then gives $e(C_{sm}) = e(C_{0,sm}) - 2\tilde{G}$. Hence, we have

$$\sum_{i\geqslant 0} e(C^{[i]})q^i = \sum_{i\geqslant 0} e(C_{sg}^{[i]})q^i(1-q)^{-e(C_{sm})} = (1-q)^{2\,\tilde{\mathrm{G}}} \sum_{i\geqslant 0} e(C_{0,sg}^{[i]})q^i(1-q)^{-e(C_{0,sm})}.$$

We consider an analogue stratification to the one in (2.4) for $C_0^{[i]}$. And, as before, we obtain

$$\sum_{i\geqslant 0} e(C_0^{[i]})q^i = \sum_{i\geqslant 0} e(C_{0,sg}^{[i]})q^i(1-q)^{-e(C_{0,sm})}.$$

Then, putting the above equations together,

$$\sum_{i\geqslant 0} e(C^{[i]})q^i = (1-q)^{2\,\tilde{\mathrm{G}}} \sum_{i\geqslant 0} e(C_0^{[i]})q^i.$$

By construction, C_0 has arithmetic genus $G - \tilde{\mathrm{G}}$. Then, by Theorem 2.6 we have

$$q^{1+\tilde{\mathrm{G}}-G} \sum_{i\geqslant 0} e(C_0^{[i]})q^i = \sum_{0\leqslant g\leqslant G-\tilde{\mathrm{G}}} n_g(C_0)F_g(q).$$

Putting everything together and using Theorem 2.6 for C, we obtain

$$\sum_{g=0}^{G} n_g(C)F_g(q) = q^{1-G}\sum_{i\geqslant 0} e(C^{[i]})q^i = (1-q)^{2\,\tilde{\mathrm{G}}}q^{-\tilde{\mathrm{G}}}\sum_{g=0}^{G-\tilde{\mathrm{G}}} n_g(C_0)F_g(q) = \sum_{g=\tilde{\mathrm{G}}}^{G} n_{g-\tilde{\mathrm{G}}}(C_0)F_g(q).$$

We conclude that $n_g(C) = 0$ for $g < \tilde{\mathrm{G}}$ and $n_g(C) = n_{g-\tilde{\mathrm{G}}}(C_0)$ for $\tilde{\mathrm{G}} \leqslant g \leqslant G$. □

We now present two examples to illustrate that the numbers $n_g(C)$ depend on the singularities of C, and that while in some cases $n_g(C)$ precisely counts partial normalisations of C, in general this holds only up to a multiplicity that depends on the singularities of C.

Example 2.8. Let C be a rational curve with one nodal singularity. Then, $n_0(C) = 1$ and $n_1(C) = 1$.

Proof. We have

$$\sum_{k\geqslant 0} e(C^{[k]})q^k = n_0F_0(q) + n_1.$$

Then, $n_1 = a_0 = e(C^{[0]}) = 1$ and $n_0 = c = e(\overline{\mathrm{Pic}^0}(C)) = 1$. □

Example 2.9. Let C be a rational curve with one cuspidal singularity. Then, $n_0(C) = 2$ and $n_1(C) = 1$.

Proof. As in the previous example, we have $n_1 = e(C^{[0]}) = 1$ and $n_0 = e(\overline{\mathrm{Pic}^0}(C))$. Since cuspidal singularities are of type A_2, we have $e(\overline{\mathrm{Pic}^0}(C)) = 2$, see Appendix 2. □

Note that in Example 2.8, the numbers $n_g(C)$ explicitly count partial normalisations of C. We now show that this holds for any nodal curve.

Lemma 2.10. *Let the multiplicative group $\mathbb{G}_m$ act on a scheme of finite type. Then, we have $e(X) = e(X^{\mathbb{G}_m})$. In particular, if the action has no fixed points, $e(X) = 0$.*

Proof. See Corollary 2 in [Bia73]. □

Theorem 2.11 ([KST11]). *Let C be a nodal curve of arithmetic genus G and geometric genus $\tilde{G}$. In particular, C has $G - \tilde{G}$ nodes. Then,*

$$e(C^{[k]}) = \sum_{0 \leqslant j \leqslant k} \binom{G - \tilde{G}}{j} e(C^{(k-j)}_{G-j,sm}),$$

where C_{G-j} denotes any partial normalisation of C at j nodes and $C_{G-j,sm}$ denotes its smooth locus.

Proof. Let $\Sigma \subset C_{sg}$ be a subset of the singular locus of C. Denote by $\pi_\Sigma \colon C_\Sigma \longrightarrow C$ the partial normalisation of C at Σ (desingularisation over Σ) and by $C_{\Sigma,sm}$ its smooth locus.
By smoothness, $C^{(i)}_{\Sigma,sm} = C^{[i]}_{\Sigma,sm}$ for all i. Define $i_\Sigma \colon C^{(k-|\Sigma|)}_{\Sigma,sm} \longrightarrow C^{[k]}$ in the following form. Given an effective Cartier divisor $Z = Z_1 \cup Z_2 \subset C_{\Sigma,sm}$, where $Z_1 := Z \cap (C_{\Sigma,sm} - \pi_\Sigma^{-1}(\Sigma))$ and $Z_2 := Z \cap \pi_\Sigma^{-1}(\Sigma)$, we pushforward Z_1 along the isomorphism $C - \Sigma \simeq C_\Sigma - \pi_\Sigma^{-1}(\Sigma)$ induced by the partial normalisation $\pi_\Sigma \colon C_\Sigma \longrightarrow C$.
Let us now describe how to attach the contribution from Z_2. Let $p \in \Sigma \subset C$ be a node, then the local model of C around p is given by $\mathbb{C}[[x,y]]/(xy)$ and its normalisation has two local branches corresponding to the x- and y-axes. If Z_2 has multiplicities a and b along the x and y branches respectively. we push Z_2 down around p to the length $a+b+1$ subscheme with local ideal (x^{a+1}, y^{b+1}) (i.e. we thicken p further by the corresponding multiplicities of Z_2 on x and y). We repeat this process with all the nodes $p \in \Sigma$. Note that if $Z_2 = \varnothing$, we attach each node $p \in \Sigma$ as a length 1 contribution. This construction yields a closed immersion $i_\Sigma \colon C^{(k-|\Sigma|)}_{\Sigma,sm} \hookrightarrow C^{[k]}$. Note that each i_Σ produces a closed subschemes whose support contains Σ and does not contain $C_{sg} - \Sigma$. Thus, the images of the morphisms i_Σ are disjoint and we obtain

$$\bigsqcup_{\Sigma \subset C_{sg}} C^{(k-|\Sigma|)}_{\Sigma,sm} \subset C^{[k]}.$$

The divisors that do not intersect the singular locus C_{sg} are obtained via the above construction applied to $\Sigma = \varnothing$. On the other hand, for $\Sigma \neq \varnothing$, the divisors obtained via the above described pushforward along i_Σ are not Cartier. Thus, the construction misses precisely those points that correspond to Cartier divisors on C and that intersect the singular locus C_{sg}. However, we claim that those divisors do not contribute to the Euler characteristic of $C^{[k]}$.
Let $p_i \in C_{sg}$ be a node, so the local model of C around p_i is given by $\mathbb{C}[[x,y]]/(xy)$. The effective Cartier divisors of C that meet C_{sg} have the form $(h_i(x), g_i(y))$ around p_i, where $h_i(x) \in \mathbb{C}[[x]]^*$ and $g_i(y) \in \mathbb{C}[[y]]^*$ and $h_i(0) = g_i(0)$. Consider the $\mathbb{C}^*$-action on the set of effective Cartier divisors that meet the singular locus C_{sg} given by

$$t \cdot (h_i(x), g_i(y))_{C_{sg}} \longmapsto (h_i(tx), g_i(y))_{C_{sg}}.$$

At the level of formal series, this action has as fixed points the elements with constant $h_i(x)$. Hence, at the level of Cartier divisors, it has no fixed points. Then, via Lemma 2.10, we have $e(C^{[k]} - \bigsqcup_{\Sigma \subset C_{sg}} C^{(k-j)}_{\Sigma,sm}) = 0$.

Thus, the stratification defined previously yields

$$e(C^{[k]}) = \sum_{0\leqslant j\leqslant k} \sum_{\substack{\Sigma\subset C_{sg}\\ |\Sigma|=j}} e(C^{(k-j)}_{\Sigma,sm}) = \sum_{0\leqslant j\leqslant k} \binom{G-\tilde{\mathrm{G}}}{j} e(C^{(k-j)}_{G-j,sm}),$$

where $C_{G-j,sm}$ denotes the smooth locus of any partial normalisation C_{G-j} of C at j nodes. □

Corollary 2.12. *Let C be a nodal curve of arithmetic genus G and geometric genus $\tilde{\mathrm{G}}$. In particular, C has $G-\tilde{\mathrm{G}}$ nodes. Then,*

$$n_i(C) = \binom{G-\tilde{\mathrm{G}}}{G-i}$$

is the number of partial normalisations of C at subsets $\Sigma\subset C_{sg}$ of cardinality $G-i$.

Proof. By comparing coefficients on both sides of equation (2.3), we conclude that the integers $n_k(C)$ satisfy the following relation (this result does not require C nodal):

$$n_{G-k}(C) = e(C^{[k]}) - \sum_{G-k+1\leqslant i\leqslant G} n_i(C) e(X_i^{(k-G+i)}), \tag{2.5}$$

where X_i denotes any smooth curve of genus i. From this relation follows $n_G(C) = 1$. Assume that the claimed result holds for $i > G-k$. By Theorem 2.11, we have

$$e(C^{[k]}) = \sum_{0\leqslant j\leqslant k} \binom{G-\tilde{\mathrm{G}}}{j} e(C^{(k-j)}_{G-j,sm}),$$

where $C_{G-j,sm}$ denotes the smooth locus of any partial normalisation C_{G-j} of C at j nodes. Note that $e(C^{(k-j)}_{G-j,sm}) = e(X^{(k-j)}_{G-j})$, where X_{G-j} denotes any smooth curve of genus $G-j$. Then, the induction hypothesis applied to (2.5) yields

$$n_{G-k}(C) = \binom{G-\tilde{\mathrm{G}}}{k} e(X^{(0)}_{G-k}) = \binom{G-\tilde{\mathrm{G}}}{k}.$$

□

3. Stable pair and BPS invariants

In this section, we present the BPS invariants for the total moduli space of stable pairs on a K3 surface following the approach by Kawai and Yoshioka, [KY00]. The setting of this section is the following. Let (X,H) be a polarised K3 surface and let $C_G\subset X$ be a curve with $C_G^2 = 2G-2$ such that $C_G\cdot H = \min\{\mathcal{L}\cdot H > 0 : \mathcal{L}\in \mathrm{Pic}(X)\}$. We call the later condition *condition of minimal intersection.*

Under the above assumptions, in Subsection 3.1 we prove following Yau–Zaslow's type generating function for the total moduli of stable pairs

$$\sum_{G\geqslant 0}\sum_{d\geqslant 0} e(P_{d+1-G}(X,[C_G]))q^{G-1}y^{d+1-G} = \frac{(y^{-1/2}-y^{1/2})^{-2}}{q\prod_{n\geqslant 1}(1-q^n)^{20}(1-q^ny)^2(1-q^ny^{-1})^2}, \tag{3.1}$$

where $e(P_{d+1-G}(X,[C_G]))$ denotes the topological Euler characteristic of the moduli space of stable pairs of curve class $[C_G]$ and Euler characteristic $d+1-G$. The concept of stable pair is introduced later in this section. The subsection 3.1 is based on [KY00], however we incorporate modified versions of some proofs of Yoshioka, see Lemmas 3.3 and 3.4. These modifications are essential for extending the results to our setting.

In Subsection 3.2, we use our calculations from Section 2 to prove the relation

$$\sum_{G\geqslant 0}\sum_{k\geqslant 1-G} e(P_k(X,[C_G]))y^k q^G = \sum_{G\geqslant 0}\sum_{g\geqslant 0} N_{g,G} y^{1-g}(1-y)^{2g-2}q^G,$$

where $N_{g,G}$ are integers. This relation, together with equation (3.1), allow us to recover the usual Yau–Zaslow's formula from equation (1.1) in the limit $y \longrightarrow 1$. This gives us an interpretation of the BPS invariants $N_{0,G}$.

Finally, in Subsection 3.3 we explore the contributions of curves in the linear system $|C_G|$ to the BPS invariants $N_{g,G}$ and relate them to the integers $n_g(C)$ introduced in Section 2. This allows us to conclude $N_{g,G} = 0$ for $g > G$.

3.1. **The generating function for stable pairs.** Let (X, H) be a polarised K3 surface and let $C_G \subset X$ be a curve with $C_G^2 = 2G - 2$ satisfying the condition of minimal intersection introduced at the beginning of the section.

Remark 3.1. Let $C \subset X$ be a curve satisfying the condition of minimal intersection. Then, any element in $|C|$ is integral. Furthermore, C is primitive. Indeed, assume there exists non-integral $D \in |C|$, then we can write $D = \sum_i a_i D_i$ for D_i integral. Since D is effective, we obtain a contradiction to the condition of minimal intersection.

We denote by $\mathcal{M}(r, [C], a)$ the *moduli space of μ-(semi)stable sheaves* on X with respect to the fixed polarisation H, of rank r, first Chern class $[C]$ and Euler characteristic a, see Appendix 1. In this section, we assume that all the spaces moduli $\mathcal{M}(r, [C], a)$ are moduli of μ-stable sheaves. By the condition of minimal intersection on C, for a general polarisation H, $\mathcal{M}(r, [C], a)$ is a moduli of μ-stable sheaves, see Theorem 4.C.3. in [HL10].

Given a Mukai vector v, consider the moduli functor

$$\mathrm{Syst}^n(v)(S) := \{q^*\mathcal{L} \longrightarrow \mathcal{F} : \mathcal{F} \in \mathcal{M}(v)(S), \mathcal{L} \text{ locally free sheaf of rank } n \text{ on } S\}/\simeq,$$

where $q\colon X \times S \longrightarrow S$ denotes the projection. This corresponds to the coarse *moduli space of coherent systems* introduced by Le Potier, see [Le 93], under a choice of stability condition for which a coherent system $q^*\mathcal{L} \longrightarrow \mathcal{F}$ is stable if, and only if $\mathcal{F}$ is stable. Note that for $S = \mathbb{C}$, a choice $q^*\mathcal{L} \longrightarrow \mathcal{F}$ corresponds to a choice of subspace $U \subset H^0(\mathcal{F})$ with $\dim(U) = n$.

Theorem 3.2 ([KY00])**.** *If $C \subset X$ satisfies the condition of minimal intersection,* $\mathrm{Syst}^n(r, [C], a)$ *is a smooth scheme of dimension $\langle v, v\rangle + 2 - n(n + \langle v(\mathcal{O}_X), v\rangle)$.*

Proof. It was proved in [He98] that the tangent space at $\Lambda := (U \otimes \mathcal{O}_X \longrightarrow \mathcal{F}) \in \mathrm{Syst}^n(r, [C], a)$ is given by $\mathbb{E}xt^1(\Lambda, \Lambda)$ and obstructions of infinitesimal liftings lie in the kernel of the following morphism

$$\tau\colon \mathbb{E}xt^2(\Lambda, \Lambda) \longrightarrow Ext^2(\mathcal{F}, \mathcal{F}) \xrightarrow{tr} H^2(X, \mathcal{O}_X).$$

So, we need to show that τ is injective. In [He98] was also shown that

$$\mathbb{E}xt^2(\Lambda,\Lambda) \simeq \mathbb{E}xt^2(\Lambda,\mathcal{F}).$$

Moreover, there is an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbb{E}xt^0(\Lambda,\Lambda) &\longrightarrow \mathrm{Hom}(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{Hom}(U\otimes\mathcal{O}_X,\mathcal{F})/V \\ \longrightarrow \mathbb{E}xt^1(\Lambda,\Lambda) &\longrightarrow \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{Ext}^1(U\otimes\mathcal{O}_X,\mathcal{F}) \\ \longrightarrow \mathbb{E}xt^2(\Lambda,\Lambda) &\longrightarrow \mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{Ext}^2(U\otimes\mathcal{O}_X,\mathcal{F}) = 0, \end{aligned}$$

where $V := \mathrm{im}(\mathrm{Hom}(U\times\mathcal{O}_X, U\times\mathcal{O}_X) \longrightarrow \mathrm{Hom}(U\times\mathcal{O}_X,\mathcal{F}))$. Then, the Serre dual of τ is given by the composition

$$H^0(X,\mathcal{O}_X) \longrightarrow \mathrm{Hom}(\mathcal{F},\mathcal{F}) \hookrightarrow \mathbb{H}om(\mathcal{F}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}). \tag{3.2}$$

We are done if we show that $\mathbb{H}om(\mathcal{F}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) \simeq \mathbb{C}$.
Let

$$0 \longrightarrow \mathcal{O}_X \otimes \mathrm{Ext}^1(\mathcal{F},\mathcal{O}_X)^\vee \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{3.3}$$

be the universal extension, i.e. the extension class corresponding to the identity element in

$$\mathrm{End}(\mathrm{Ext}^1(\mathcal{F},\mathcal{O}_X)) \simeq \mathrm{Ext}^1(\mathcal{F},\mathcal{O}_X\otimes\mathrm{Ext}^1(\mathcal{F},\mathcal{O}_X)^\vee).$$

By (3.2), we have $\dim \mathbb{H}om(\mathcal{F}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) \geqslant 1$. Hence, via the exact sequence (3.3) it is sufficient to show that $\dim \mathbb{H}om(\mathcal{G}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) = 1$ and the natural morphism

$$\mathbb{H}om(\mathcal{F}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) \longrightarrow \mathbb{H}om(\mathcal{G}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F})$$

is injective.

By Theorem 2.5. in [Yos99] and Serre duality, $\mathrm{Ext}^1(\mathcal{G},\mathcal{O}_X) = H^1(X,\mathcal{G})^\vee = 0$. By stability of $\mathcal{G}$ we have $\mathrm{Hom}(\mathcal{G},\mathcal{O}_X) = 0$. Furthermore, we have the exact sequence

$$\mathrm{Hom}(\mathcal{G}, U\otimes\mathcal{O}_X) \longrightarrow \mathrm{Hom}(\mathcal{G},\mathcal{F}) \longrightarrow \mathbb{H}om(\mathcal{G}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) \longrightarrow \mathrm{Ext}^1(\mathcal{G}, U\otimes\mathcal{O}_X),$$

hence $\mathrm{Hom}(\mathcal{G},\mathcal{F}) \simeq \mathbb{H}om(\mathcal{G}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F})$. Note that $\mathrm{Hom}(\mathcal{G},\mathcal{F})$ fits in the exact sequence

$$\mathrm{Hom}(\mathcal{G},\mathcal{O}_X^{\oplus i}) \longrightarrow \mathrm{Hom}(\mathcal{G},\mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{G},\mathcal{F}) \longrightarrow \mathrm{Ext}^1(\mathcal{G},\mathcal{O}_X^{\oplus i}),$$

where $i := \dim \mathrm{Ext}^1(\mathcal{F},\mathcal{O}_X)$. Then, we obtain $\dim \mathbb{H}om(\mathcal{G}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) = 1$ by simplicity of $\mathcal{G}$.

Consider the exact sequence

$$\mathbb{E}xt^{-1}(\mathcal{O}_X\oplus i, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) \longrightarrow \mathbb{H}om(\mathcal{F}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) \longrightarrow \mathbb{H}om(\mathcal{G}, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}).$$

We show that $\mathbb{E}xt^{-1}(\mathcal{O}_X\oplus i, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) = 0$. Note that

$$\mathbb{E}xt^{-1}(\mathcal{O}_X\oplus i, U\otimes\mathcal{O}_X \longrightarrow \mathcal{F}) = \ker(\mathrm{Hom}(\mathcal{O}_X^{\oplus i}, U\otimes\mathcal{O}_X) \longrightarrow \mathrm{Hom}(\mathcal{O}_X^{\oplus i},\mathcal{F})).$$

Since U is a subspace of $\mathrm{Hom}(\mathcal{O}_X,\mathcal{F})$, we obtain the claimed result. $\square$

In order to construct the generating function of the total moduli space of stable pairs, we need some intermediate results. Following two lemmas have been proved by Yoshioka under the assumption $\mathrm{Pic}(X) = C_G\mathbb{Z}$, cf. [Yos99]. Here, we present different proofs of the statements under a weaker assumption, i.e. C_G satisfying the minimal intersection condition.

Lemma 3.3. *Let $C_G \subset X$ be a curve of arithmetic genus G satisfying the condition of minimal intersection. Given a μ-stable sheaf $\mathcal{E}$ with $r := rk(\mathcal{E}) \geqslant 1, d := \deg(\mathcal{E}) = C_G \cdot H$ and a non-zero morphism $\phi\colon \mathcal{O}_X \longrightarrow \mathcal{E}$, we have that ϕ is injective and $\mathrm{coker}(\phi)$ is μ-semistable.*
In this section we assumed that μ-semistable sheaves are μ-stable, hence $\mathrm{coker}(\phi)$ is μ-stable.

Proof. Let us first consider the case $r > 1$. Then, ϕ is not surjective. By stability of $\mathcal{E}$, we have $0 \leqslant \deg(\mathrm{im}(\phi))/\mathrm{rk}(\mathrm{im}(\phi)) < d/r$. If $\deg(\mathrm{im}(\phi)) > 0$, by minimality of d we have $d \leqslant \deg(\mathrm{im}(\phi))$ and so $dr \leqslant \deg(\mathrm{im}(\phi))r < d\,\mathrm{rk}(\mathrm{im}(\phi))$. Then, $r < \mathrm{rk}(\mathrm{im}(\phi))$, which is a contradiction. Then, $\mu(\mathcal{O}_X/\ker(\phi)) = \mu(\mathcal{O}_X)$, which contradicts the stability of $\mathcal{O}_X$ unless ϕ is injective.
Thus, we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\phi} \mathcal{E} \longrightarrow \mathcal{F} := \mathrm{coker}(\phi) \longrightarrow 0. \tag{3.4}$$

In particular, $\deg(\mathcal{F}) = d$ and $\mathrm{rk}(\mathcal{F}) = r - 1$. Assume that $\mathcal{F}$ is not μ-semistable, then there exists a quotient sheaf of $\mathcal{F}$, say $\mathcal{F} \twoheadrightarrow \mathcal{G}$ such that $\deg(\mathcal{G})/\mathrm{rk}(\mathcal{G}) < d/(r-1)$. Since $\mathcal{E}$ surjects onto $\mathcal{F}$, then $\mathcal{G}$ is also a quotient sheaf of $\mathcal{E}$. By stability of $\mathcal{E}$ and by minimality of of d we get

$$d(r-1) < \deg(\mathcal{G})(r-1) < d\,\mathrm{rk}(\mathcal{G}),$$

and so $r - 1 < \mathrm{rk}(\mathcal{G})$, which contradicts that $\mathcal{G}$ is a quotient sheaf of $\mathcal{F}$.
For the case $r = 1$, $\phi\colon \mathcal{O}_X \longrightarrow \mathcal{E}$ is injective, otherwise its kernel destabilises $\mathcal{O}_X$, and $\mathrm{coker}(\phi)$ has rank 0. We have that $\mathrm{supp}(\mathrm{coker}(\phi)) = C$ is a curve. Since $\deg(\mathrm{coker}(\phi)) = d$ and we assumed d to be minimal among positive intersections, C must be integral. Finally, as $\mathrm{coker}(\phi)$ is a torsion free sheaf of rank 1 on C and C is integral, it is μ-stable. □

Lemma 3.4. *Let C be a curve satisfying the condition of minimal intersection. Then, if $\mathcal{F}$ is a μ-stable sheaf with $d = \deg(\mathcal{F}) = C \cdot H$ and rank $r = \mathrm{rk}(\mathcal{F}) \geqslant 1$, any non-trivial extension*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

is μ-semistable.

Proof. Assume that $\mathcal{E}$ is not μ-semistable. Then, there exists stable subsheaf $\mathcal{G} \hookrightarrow \mathcal{E}$ of degree d_G and rank r_G such that $r_G < \mathrm{rk}(\mathcal{E}) = r + 1$ and $0 \leqslant d/(r+1) < d_G/r_G$. Then, $d \leqslant d_G$ by minimality. Moreover, the composition $\phi\colon G \hookrightarrow \mathcal{E} \longrightarrow \mathcal{F}$ is non-trivial, and so the stability of $\mathcal{F}$ yields $d_G/r_G \leqslant d/r$. In particular, $r \leqslant r_G < r + 1$, so we have $r_G = r$. Hence, by minimality of d and the relation $d_G/r_G \leqslant d/r$, we obtain $d_G = d$. Since $\mathcal{F}$ and $\mathcal{G}$ are μ-stable of same slope, ϕ is an isomorphism in codimension 1.
Denote the extension class by $e \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X)$. Then, ϕ induces a morphism

$$\Phi\colon\ \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^1(\mathcal{G}, \mathcal{F}),$$

sending $e \longmapsto 0$. We have $\mathrm{Ext}^1(\mathcal{F}/\mathcal{G}, \mathcal{O}_X) = 0$, and so Φ is injective. Then, $e = 0$, which is a contradiction. □

Remark 3.5. We define the following stratification

$$\mathcal{M}(r,[C],a) = \bigsqcup_{i \geqslant r+a} \mathcal{M}(r,[C],a)_i,$$

where $\mathcal{M}(r,[C],a)_i := \{\mathcal{F} \in \mathcal{M}(r,[C],a) : h^0(\mathcal{F}) = i\}$. Analogously, we have the stratification

$$\mathrm{Syst}^1(r,[C],a) = \bigsqcup_{i \geqslant r+a} \mathrm{Syst}^1(r,[C],a)_i,$$

where $\mathrm{Syst}^1(r,[C],a)_i := \{(\mathcal{F}, s \in H^0(\mathcal{F})) : \mathcal{F} \in \mathcal{M}(r,[C],a), h^0(\mathcal{F}) = i\}$. Given a sheaf $\mathcal{F} \in \mathcal{M}(r,[C],a)$ with $i < r+a$, the condition $h^0(\mathcal{F}) = i$ does never hold because via stability $h^2(\mathcal{F}) = hom(\mathcal{F}, \mathcal{O}_X) = 0$.

Lemmas 3.3 and 3.4 allow us to prove following central result.

Theorem 3.6. *Let C satisfy the condition of minimal intersection and let $r \geqslant 1$, $v = (r,[C],a)$ and $w = (r-1,[C],a-1)$. Any element $(s\colon \mathcal{O}_X \longrightarrow \mathcal{E}) \in \mathrm{Syst}^1(v)$ is injective and* $\mathrm{coker}(s)$ *is a μ-stable sheaf. Hence, we have a morphism*

$$q_v\colon \mathrm{Syst}^1(v) \longrightarrow \mathcal{M}(w), \; s \longmapsto \mathrm{coker}(s).$$

Moreover, by setting $m = 1-(r+a)$, we obtain following diagram for $i \geqslant r+a$:

$$\begin{array}{ccc} & \mathrm{Syst}^1(v)_i & \\ {\scriptstyle p_v} \swarrow & & \searrow {\scriptstyle q_v} \\ \mathcal{M}(v)_i & & \mathcal{M}(w)_{i-1} \end{array}$$

where the forgetful morphism p_v is an étale locally trivial $\mathbb{P}^i$-bundle and q_v is an étale locally trivial $\mathbb{P}^{m+i}$-bundle.

Proof. The injectivity of any $s\colon \mathcal{O}_X \longrightarrow \mathcal{E} \in \mathrm{Syst}^1(v)$ and the stability of $\mathrm{coker}(s)$ follow from Lemma 3.3. The additivity of Mukai vectors on short exact sequences ensures $v(\mathrm{coker}(s)) = w$. Since $h^1(X) = 0$, we have $h^0(\mathrm{coker}(f)) = h^0(\mathcal{E}) - h^0(\mathcal{O}_X) = i-1$.
For the rest of the claim it is enough to check the fibres of the morphisms p_v and q_v. Clearly, $p_v^{-1}(\mathcal{E}) = \mathbb{P}(H^0(X,\mathcal{E}))$. Additionally, Lemma 3.4 yields $q_v^{-1}(\mathcal{F}) \simeq \mathbb{P}(\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X))$. Finally, $ext^1(\mathcal{F}, \mathcal{O}_X) = ext^1(\mathcal{O}_X, \mathcal{F}) = h^1(\mathcal{F}) = m+i$, as $\chi(\mathcal{F}) = \chi(\mathcal{E}) - \chi(\mathcal{O}_X) = r+a-2$, and $h^2(\mathcal{F}) = hom(\mathcal{F}, \mathcal{O}_X) = 0$ is clear for rank 0, and follows for positive rank by stability of $\mathcal{F}$. □

Definition 3.7. Given a smooth complex projective variety V, we define its *Hodge polynomial* by

$$\chi_{t,t'}(V) := \sum_{p,q=0}^{\dim(V)} (-1)^{p+q} h^{p,q}(V) t^p t'^q,$$

where $h^{p,q}(V)$ denotes the (p,q)-Hodge number of V.

Lemma 3.8. *Let $\pi\colon V \longrightarrow W$ be an étale locally trivial $\mathbb{P}^n$-bundle with V, W smooth, such that V is projective over W. Then,*

$$\chi_{t,t'}(V) = [n+1]\chi_{t,t'}(W),$$

where $[n] := \frac{(tt')^n - 1}{tt' - 1}$.
Additionally, given a decomposition $V = \bigcup_i V_i$ into mutually disjoint locally closed subsets, we have

$$\chi_{t,t'}(V) = \sum_i \chi_{t,t'}(V_i).$$

Proof. See Lemma 5.163 in [KY00]. □

Following result is the key part of Kawai–Yoshioka's construction of the claimed closed form.

Theorem 3.9 ([KY00])**.** *Let $C_G \subset X$ be a curve of arithmetic genus G satisfying the condition of minimal intersection. Then, for $r + a \geqslant 0$, we have the decomposition*

$$\chi_{t,t'}(\mathrm{Syst}^1(r, [C_G], a)) = \sum_{k \geqslant 0} (tt')^{(r+a+k-1)k}[r + a + 2k]\chi_{t,t'}(\mathcal{M}(r + k, [C_G], a + k)).$$

Note that the sum on the right hand side is not infinite as $\mathcal{M}(r + k, [C_G], a + k) = \varnothing$ for $G - (r + k)(a + k) < 0$.

Proof. Via Lemma 3.6, we have the diagram:
(3.5)

$$\mathcal{M}(r+1, [C_G], a+1)_{r+a+1+i} \xleftarrow{\ p\ } \mathrm{Syst}^1(r+1, [C_G], a+1)_{r+a+1+i} \xrightarrow{\ q\ } \mathcal{M}(r, [C_G], a)_{r+a+i},$$

where p is a $\mathbb{P}^{r+a+i}$ étale bundle and q is a $\mathbb{P}^{i-1}$ étale bundle.
This diagram and Lemma 3.8 yield

$$\begin{aligned}
\sum_{i \geqslant 0} [i]\chi_{t,t'}(\mathcal{M}(r, [C_G], a)_{r+a+i}) &= \sum_{i \geqslant 0} \chi_{t,t'}(\mathrm{Syst}^1(r + 1, [C_G], a + 1)_{r+a+1+i}) \\
&= \sum_{i \geqslant 0} [r + a + 1 + i]\chi_{t,t'}(\mathcal{M}(r + 1, [C_G], a + 1)_{r+a+1+i}) \\
&= \sum_{i \geqslant 0} [r + a + 2 + i]\chi_{t,t'}(\mathcal{M}(r + 1, [C_G], a + 1)_{r+a+2+i}).
\end{aligned}$$

We have $(tt')^{r+a+2}[i] + [r + a + 2] = [r + a + 2 + i]$, hence via the stratification of $\mathcal{M}(v)$ with $v = (r + 1, [C_G], a + 1)$, we can write

$$\begin{aligned}
(tt')^{r+a+2} \sum_{i \geqslant 0} [i]\chi_{t,t'}(\mathcal{M}(v)_{r+a+2+i}) &= \sum_{i \geqslant 0} ([r + a + 2 + i] - [r + a + 2])\chi_{t,t'}(\mathcal{M}(v)_{r+a+2+i}) \\
&= \sum_{i \geqslant 0} [r + a + 2 + i]\chi_{t,t'}(\mathcal{M}(v)_{r+a+2+i}) - [r + a + 2]\chi_{t,t'}(\mathcal{M}(v)).
\end{aligned}$$

Putting the last two equations together we obtain

$$\begin{aligned}\sum_{i\geqslant 0}[i]\chi_{t,t'}(\mathcal{M}(r,[C_G],a)_{r+a+i}) &= \sum_{i\geqslant 0}[r+a+2+i]\chi_{t,t'}(\mathcal{M}(v)_{r+a+2+i})\\ &= (tt')^{r+a+2}\sum_{i\geqslant 0}[i]\chi_{t,t'}(\mathcal{M}(v)_{r+a+2+i}) + [r+a+2]\chi_{t,t'}(\mathcal{M}(v)),\end{aligned}$$

where $v=(r+1,[C_G],a+1)$. Doing the above calculation inductively, we have the following relation

(3.6)
$$\sum_{i\geqslant 0}[i]\chi_{t,t'}(\mathcal{M}(r,[C_G],a)_{r+a+i}) = \sum_{k\geqslant 1}(tt')^{\sum_{j=1}^{k-1}(r+a+2j)}[r+a+2k]\chi_{t,t'}(\mathcal{M}(r+k,[C_G],a+k)).$$

The additivity of the Hodge polynomial and Lemma 3.8 applied to diagram (3.5) imply

$$\chi_{t,t'}(\mathrm{Syst}^1(r,[C_G],a)) = \sum_{i\geqslant 0}[r+a+i]\chi_{t,t'}(\mathcal{M}(r,[C_G],a)_{r+a+i}).$$

We note that $(tt')^{r+a}[i]+[r+a]=[r+a+i]$. Hence, as before, we can write

$$\chi_{t,t'}(\mathrm{Syst}^1(r,[C_G],a)) = (tt')^{r+a}\sum_{i\geqslant 0}[i]\chi_{t,t'}(\mathcal{M}(r,[C_G],a)_{r+a+i}) + [r+a]\mathcal{M}(r,[C_G],a).$$

Via equation (3.6), we obtain

$$\chi_{t,t'}(\mathrm{Syst}^1(r,[C_G],a)) = \sum_{k\geqslant 0}(tt')^{(r+a+k-1)k}[r+a+2k]\chi_{t,t'}(\mathcal{M}(r+k,[C_G],a+k)),$$

where we used that $\sum_{0\leqslant j\leqslant k-1}(r+a+2j)=(r+a+k-1)k$. □

Theorem 3.10. *Let $C_G\subset X$ be a curve satisfying the condition of minimal intersection. Then, $\mathcal{M}(r,[C_G],a)$ is deformation equivalent to $X^{[G-ra]}$. In particular, we have*

$$\chi_{tt'}(\mathcal{M}(r,[C_G],a)) = \chi_{tt'}(X^{[G-ra]}).$$

Proof. This result has been proved by Yoshioka for the case $r>0$, see Theorem 7.4. A modification of Yoshioka's proof for the case $r=0$ can be found in Theorem 7.1. □

Given a section $s\colon \mathcal{O}_X\longrightarrow\mathcal{F}$, we obtain the distinguished triangle

$$\mathcal{O}_X \xrightarrow{s} \mathcal{F}\longrightarrow[\mathcal{O}_X\longrightarrow\mathcal{F}] = C(s),$$

which yields the exact sequence

(3.7) $$0\longrightarrow\mathcal{H}om(C(s),\mathcal{O}_X)\longrightarrow\mathcal{H}om(\mathcal{F},\mathcal{O}_X)\longrightarrow\mathcal{H}om(\mathcal{O}_X,\mathcal{O}_X)\xrightarrow{t}\mathcal{E}xt^1(C(s),\mathcal{O}_X)\longrightarrow\dots.$$

This allows us to define the following morphism.

Theorem 3.11 ([KY00]). *Let $C \subset X$ satisfy the condition of minimal intersection. Then, we have an isomorphism*

$$\phi\colon \operatorname{Syst}^1(0,[C],a) \xrightarrow{\simeq} \operatorname{Syst}^1(1,[C],1-a),\ (\mathcal{O}_X \xrightarrow{s} \mathcal{F}) \longmapsto (\mathcal{O}_X \xrightarrow{t} \mathcal{E}xt^1(C(s),\mathcal{O}_X)).$$

Furthermore, this induces following isomorphism at the level of the stratification

$$\operatorname{Syst}^1(0,[C],a)_{a+i} \simeq \operatorname{Syst}^1(1,[C],1-a)_{1+i}.$$

Proof. See Proposition 5.128 in [KY00]. □

Corollary 3.12. *Under the conditions of the previous theorem, we have the following diagram*

$$\begin{array}{ccc} & \operatorname{Syst}^1(0,[C],a)_{a+i} \simeq \operatorname{Syst}^1(1,[C],1-a)_{1+i} & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ \mathcal{M}(0,[C],a)_{a+i} & & \mathcal{M}(1,[C],1-a)_{1+i} \end{array},$$

where the forgetful morphisms p_1 and p_2 are étale locally trivial $\mathbb{P}^{a+i}$-, respectively $\mathbb{P}^{1+i}$-bundles.

Theorem 3.13 ([KY00]). *Assume that $C_G \subset X$ satisfies the condition of minimal intersection for all $G \geqslant 0$. Then, for $|q| < |y| < 1$ holds*

$$\sum_{G\geqslant 0}\sum_{d\geqslant 0} \chi_{t,t'}(\operatorname{Syst}^1(0,[C_G],d+1-G))(tt')^{1-G}q^{G-1}y^{d+1-G}$$
$$= \frac{-1}{q(y)_\infty(q/y)_\infty((tt'y)^{-1})_\infty(tt'yq)_\infty(t(t')^{-1}q)_\infty(q)^{18}_\infty(t^{-1}t'q)_\infty},$$

where $(\zeta)_\infty := \prod_{n\geqslant 0}(1-\zeta q^n)$. In particular, by setting $t = t' = 1$, we obtain:

(3.8)
$$\sum_{G\geqslant 0}\sum_{d\geqslant 0} e(\operatorname{Syst}^1(0,[C_G],d+1-G))q^{G-1}y^{d+1-G} = \frac{(y^{1/2}-y^{-1/2})^{-2}}{q\prod_{n\geqslant 1}(1-q^n)^{20}(1-q^ny)^2(1-q^ny^{-1})^2},$$

where $e(-)$ denotes the topological Euler characteristic.

Proof. Let us first assume $a \geqslant 0$. Apply Theorem 3.9 for the case $r = 0$. Then,

(3.9)
$$\begin{aligned}\sum_{\substack{h\geqslant 0,\\ a\geqslant 0}} \chi_{t,t'}(\operatorname{Syst}^1(0,[C_G],a))y^a(tt')^{1-h}q^{h-1} &= \sum_{\substack{h\geqslant 0,\\ a\geqslant 0,\\ k\geqslant 0}} (tt')^{(a+k-1)k}[a+2k]\chi_{t,t'}(\mathcal{M}(k,[C_G],a+k))y^a(tt')^{1-h}q^{h-1}\\ &= \sum_{\substack{h\geqslant 0,\\ j\geqslant i,\\ i\geqslant 0}} (tt')^{(j-1)i}[i+j]\chi_{t,t'}(\mathcal{M}(i,[C_G],j))y^{j-i}(tt')^{1-h}q^{h-1}\\ &= \sum_{\substack{h\geqslant 0,\\ j\geqslant i,\\ i\geqslant 0}} (tt')^{(j-1)i}[i+j]\chi_{t,t'}(X^{[C_G^2/2-ij+1]})y^{j-i}(tt')^{1-h}q^{h-1}\\ &= \frac{tt'}{q}\Big(\sum_{j\geqslant i}\sum_{i\geqslant 0}(tt')^{-i}[i+j]y^{j-i}q^{ij}\Big)\Big(\sum_n \chi_{t,t'}(X^{[n]})(tt')^{-n}q^n\Big),\end{aligned}$$

where we used Theorem 3.10 in the third equality. For $a > 0$, Corollary 3.12 yields

$$\sum_{i\geqslant 1}[i]\chi_{t,t'}(\mathcal{M}(0,[C_G],-a)_i) = \sum_{i\geqslant 1}[a+i+1]\chi_{t,t'}(\mathcal{M}(1,[C_G],1+a)_{a+i+1}).$$

By equation (3.6) and performing a similar calculation as in equation (3.9), for $a > 0$, we obtain (3.10)

$$\sum_{\substack{h\geqslant 0,\\ a\geqslant 0}} \chi_{t,t'}(\mathrm{Syst}^1(0,[C_G],-a))y^{-a}(tt')^{1-h}q^{h-1} = \frac{tt'}{q}\Big(\sum_{i>j}\sum_{j\geqslant 1}(tt')^{-i}[i+j]y^{j-i}q^{ij}\Big)\Big(\sum_n \chi_{t,t'}(X^{[n]})(tt')^{-n}q^n\Big).$$

Combining Equations (3.9) and (3.10), we have

$$\sum_{\substack{h\geqslant 0,\\ a\in\mathbb{Z}}} \chi_{t,t'}(\mathrm{Syst}^1(0,[C_G],a))y^{a}(tt')^{1-h}q^{h-1} = \frac{tt'}{q}\Big(\sum_{\substack{i\geqslant 0,\\ j>0}}(tt')^{-i}[i+j]y^{j-i}q^{ij}\Big)\Big(\sum_n \chi_{t,t'}(X^{[n]})(tt')^{-n}q^n\Big).$$

The following identity has been proved in [Hic88]:

$$\begin{aligned}\frac{tt'}{q}\Big(\sum_{\substack{i\geqslant 0,\\ j>0}}(tt')^{-i}[i+j]y^{j-i}q^{ij}\Big) &= \frac{tt'}{q(tt'-1)}\sum_{\substack{i\geqslant 0,\\ j>0}}((tt'y)^j y^{-i} - (tt'y)^{-i}y^j)q^{ij}\\ &= \frac{-(q)_\infty^2((tt')^{-1}q)_\infty(tt'q)_\infty}{q(y)_\infty(q/y)_\infty((tt'y)^{-1})_\infty(tt'yq)_\infty},\end{aligned}$$

where $(\zeta)_\infty := \prod_{n\geqslant 0}(1-\zeta q^n)$. Finally, it has been proved that, see [Che96] and [GS93],

$$\sum_n \chi_{t,t'}(X^{[n]})(tt')^{-n}q^n = \frac{1}{((tt')^{-1}q)_\infty(t(t')^{-1}q)_\infty(q)_\infty^{20}(t^{-1}t'q)_\infty(tt'q)_\infty}.$$

□

Now, let us explore the relation between the moduli space of coherent systems $\mathrm{Syst}^1(0,C,a)$ and the moduli space of stable pairs.

Definition 3.14 (Stable pairs)**.** A pair $(\mathcal{F}, s)$ consisting of a coherent sheaf $\mathcal{F}$ on X supported in dimension 1 together with a section $s \in H^0(X,\mathcal{F})$ is called *stable pair* if $\mathcal{F}$ is a pure sheaf and $\mathrm{coker}(s\colon \mathcal{O}_X \longrightarrow \mathcal{F})$ has dimension 0.
Given by an integral curve $i\colon C \hookrightarrow X$ and a divisor $D \subset C$, we obtain the typical example of an stable pair via $(i_*\mathcal{O}_C(D), s_D)$, where s_D is the canonical section associated to D.

Lemma 3.15 ([PT10])**.** *An stable pair supported on a Gorenstein curve C is equivalent to a 0-dimension subscheme of C. Under this equivalence, the pair*

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{s} \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0$$

is associated to the subscheme

$$\mathcal{O}_C \simeq \mathcal{E}xt^0(\mathcal{O}_C,\mathcal{O}_C) \longrightarrow \mathcal{E}xt^1(\mathcal{Q},\mathcal{O}_C) \longrightarrow 0$$

Proof. The key point is the following equivalence: given a generically locally trivial sheaf $\mathcal{F}$ on a Gorenstein curve C, $\mathcal{F}$ is pure if, and only if $\mathcal{E}xt^i_C(\mathcal{F},\mathcal{O}_C) = 0$ for $i > 0$, see Appendix B in [PT10].

Let $0 \longrightarrow \mathcal{O}_C \xrightarrow{s} \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0$ be a stable pair. Then, by purity of $\mathcal{F}$, we have the exact sequence

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}_C) \longrightarrow 0.$$

Hence, $\mathcal{F}^\vee$ is an ideal sheaf and $\mathcal{E}xt^1(\mathcal{Q}, \mathcal{O}_C)$ is isomorphic to the structure sheaf of a subscheme of C [1].

On the other hand, let $D \in C$ be a divisor. Then, by purity of $\mathcal{O}_C$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{I}_D^\vee \longrightarrow \mathcal{E}xt^1(\mathcal{O}_D, \mathcal{O}_C) \longrightarrow 0,$$

where $\mathcal{I}_D$ is the ideal sheaf associated to D. Moreover, $\mathcal{E}xt^i(\mathcal{I}_D, \mathcal{O}_C) = 0$ for $i > 0$. Thus, $\mathcal{I}_D^\vee = R\mathcal{H}om(\mathcal{I}_D, \mathcal{O}_C)$, and we note that

$$R\mathcal{H}om(\mathcal{I}_D^\vee, \mathcal{O}_D) = \mathcal{I}_D.$$

This implies $\mathcal{E}xt^i(\mathcal{I}_D^\vee, \mathcal{O}_C) = 0$ for $i > 0$, and so $\mathcal{I}_D$ is pure. Hence, $\mathcal{O}_C \longrightarrow \mathcal{I}_D^\vee$ determines an stable pair. □

Given a curve class $[C] \in H_2(X, \mathbb{Z})$, we denote by $P_a(X, [C])$ the *moduli space of stable pairs* $(\mathcal{F}, s)$ with Euler characteristic $\chi(\mathcal{F}) = a$ and $c_1(\mathcal{F}) = [C]$. The following lemma relates the moduli space of coherent systems $\mathrm{Syst}^1(0, [C], a)$ and the moduli space of stable pairs $P_a(X, [C])$.

Lemma 3.16. *Let $C \subset X$ be a curve satisfying the condition of minimal intersection. Then, elements $(\mathcal{F}, s) \in \mathrm{Syst}^1(0, [C], a)$ are equivalent to stable pairs on X with support in $|C|$. In particular,* $\mathrm{Syst}^1(0, [C], a) = P_a(X, [C])$.

Proof. Let $(\mathcal{F}, s)$ be a stable pair on X such that $supp(\mathcal{F}) \in |C|$. Since C satisfies the condition of minimal intersection, $\mathrm{supp}(\mathcal{F})$ is integral. We have that $\mathrm{coker}(\mathcal{O}_X \xrightarrow{s} \mathcal{F})$ has dimension 0, so $\mathcal{F}$ is generically isomorphic to $\mathcal{O}_{\mathrm{supp}(\mathcal{F})}$. In particular, $\mathcal{F}$ has rank 1 on its support. Finally, pure sheaves of rank 1 on integral curves are μ-stable.

Let now $(\mathcal{F}, s \in H^0(\mathcal{F})) \in \mathrm{Syst}^1(0, [C], a)$. We only need to verify that $\mathrm{coker}(s)$ has dimension 0. As $\mathcal{F}$ is supported on a curve, $\mathrm{coker}(s)$ is supported on dimension 1 or 0. Assume that $\mathrm{coker}(s)$ is supported in dimension 1, then it is supported on the whole C by integrality, and it has positive rank on C. As $c_1(\mathcal{F}) = [C]$, we have $\mathcal{F}$ has rank 1 on C [2]. Then, since $\mathrm{coker}(s)$ has positive rank on C and it is a quotient sheaf of $\mathcal{F}$, $\mathrm{coker}(s)$ has rank 1 on C. This contradicts the stability of $\mathcal{F}$. □

The previous lemma allows us to write the result of Theorem 3.13 in the following form. Given a family of curves $\{C_G \subset X\}_{G \geqslant 0}$ of arithmetic genus G satisfying the condition of minimal intersection, we have

$$\sum_{G \geqslant 0} \sum_{d \geqslant 0} e(P_{d+1-G}(X, [C_G])) q^{G-1} y^{d+1-G} = \frac{(y^{-1/2} - y^{1/2})^{-2}}{q \prod_{n \geqslant 1} (1-q^n)^{20} (1-q^n y)^2 (1-q^n y^{-1})^2}.$$

[1] Note that $\mathcal{Q}$ may not be isomorphic to the structure sheaf of a subscheme of C.

[2] As $\mathcal{F}$ is torsion free on C and C is integral, $c_1(\mathcal{F}) = r[C]$ with $r = \mathrm{length}_{\mathcal{O}_{C,\eta}}(\mathcal{F}_\eta)$ for η the generic point of C

3.2. **Recovering the Yau–Zaslow formula.** In this subsection, we show the following relation for a family of curves $\{C_G \subset X\}_{G\geqslant 0}$ satisfying the condition of minimal intersection and such that $C_G^2 = 2G-2$:

$$\sum_{G\geqslant 0}\sum_{k\geqslant 1-G} e(P_k(X,[C_G]))y^k q^G = \sum_{G\geqslant 0}\sum_{l\geqslant 0} N_{l,G}y^{1-l}(1-y)^{2l-2}q^G. \tag{3.11}$$

We present here an approach that makes evident the contributions of elements of the linear system $|C_G|$ to the BPS invariants $N_{l,G}$.

Recall that we proved in Lemma 2.4 that given a Gorenstein curve C of arithmetic genus G, we have

$$e(C^{[k+G-1]}) - e(C^{[-k+G-1]}) = k\cdot e(\overline{\mathrm{Pic}^0}(C)), \tag{3.12}$$

where $\overline{\mathrm{Pic}^0}(C)$ denotes the compactified Jacobian of C.

Theorem 3.17. *Let $C_G \subset X$ be a curve of arithmetic genus G satisfying the condition of minimal intersection. Then, we have:*

$$e(P_k(X,[C_G])) - e(P_{-k}(X,[C_G])) = k\cdot\alpha,$$

where α is a constant. In particular, there exist $N_{l,G}\in\mathbb{Z}$ such that

$$\sum_{G\geqslant 0}\sum_{k\geqslant 1-G} e(P_k(X,[C_G]))y^k q^G = \sum_{G\geqslant 0}\sum_{l\geqslant 0} N_{l,G}y^{1-l}(1-y)^{2l-2}q^G.$$

Proof. By Lemma 3.15, any stable pair $(\mathcal{F},s)$ is determined by $\mathrm{supp}(\mathcal{F})$ and $\mathrm{supp}(\mathrm{coker}(s))$. Hence, pairs in $P_k(X,[C_G])$ can be thought of as pairs (C,D) for $C\in|C_G|$ and $D\in C^{[k+G-1]}$, cf. Theorem 3.19. Let $P_k(X,[C_G])\longrightarrow|C_G|$ be the morphism given by $(C,D)\longmapsto C$. For fixed $k\geqslant 0$, consider the stratification

$$|C_G| = \bigsqcup_{n\in\mathbb{Z}} T_n, \text{ where } T_n := \{C\in|C_G| : e(C^{[k+G-1]}) = n\},$$

from which we obtain

$$e(P_k(X,[C_G])) = \sum_n n\cdot e(T_n).$$

Note that $e(\overline{\mathrm{Pic}^0}(C))$ may not be constant for $C\in T_n$, as it depends on the singularities of C. Then, we stratify further

$$T_n = \bigsqcup_i T_{n,i}, \text{ where } T_{n,i} := \{C\in T_n : e(\overline{\mathrm{Pic}^0}(C)) = i\}.$$

By equation (3.12), we have

$$e(P_k(X,[C_G])) - e(P_{-k}(X,[C_G])) = k\sum_n\sum_i i\cdot e(T_{n,i}) = k\sum_i ie(\{C\in|C_G| : e(\overline{\mathrm{Pic}^0}(C)) = i\}).$$

Note that $\sum_i ie(\{C \in |C_G| : e(\overline{\mathrm{Pic}^0}(C)) = i\})$ is independent of k, and so it corresponds to the claimed constant α. Hence, Lemma 2.5 implies

$$\sum_{G\geqslant 0}\sum_{k\geqslant 1-G} e(P_k(X,[C_G]))y^k q^G = \sum_{G\geqslant 0}\sum_{l\geqslant 0} N_{l,G}y^{1-l}(1-y)^{2l-2}q^G,$$

for $N_{l,G} \in \mathbb{Z}$. □

The previous result yields

$$\sum_{G\geqslant 0}\sum_{l\geqslant 0} N_{l,G}y^{-l}(1-y)^{2l}q^G = \frac{1}{\prod_{n\geqslant 1}(1-q^n)^{20}(1-q^n y)^2(1-q^n y^{-1})^2}.$$

In particular, in the limit $y \longrightarrow 1$ we obtain (for $l \neq 0$ the y-terms on the left hand side vanish):

$$\sum_{G\geqslant 0} N_{0,G}q^G = \frac{1}{\prod_{n\geqslant 1}(1-q^n)^{24}}.$$

Hence, we recover the Yau–Zaslow formula from equation (1.1). This allows us to interpret the integers $N_{0,G}$ as the number of rational curves (counted with multiplicities) in a linear system $|C|$ of dimension G and of curves of arithmetic genus G on a K3 surface, c.f. Section 1. In particular, $N_{0,G} > 0$.

Remark 3.18. Consider the morphism

$$\phi\colon \mathcal{M}(0,[C_G],k) \longrightarrow |C_G|,\ \mathcal{F} \longmapsto \mathrm{supp}(\mathcal{F}).$$

Any torsion free sheaf of rank 1 supported on $C \in |C_G|$ is pure as C does not have embedded points. Even more, since C_G satisfies the condition of minimal intersection, we have seen that any such sheaf is stable. Hence, the fibres of ϕ are given by $\phi^{-1}(C) = \overline{\mathrm{Pic}^{k+G-1}}(C) \simeq \overline{\mathrm{Pic}^0}(C)$. Then, we obtain

$$e(\mathcal{M}(0,[C_G],k)) = \sum_i ie(\{C \in |C_G| : e(\overline{Pic^0}(C)) = i\}).$$

In particular, by the proofs of Theorem 3.17 and Lemma 2.5, we conclude

$$e(\mathcal{M}(0,[C_G],k)) = N_{0,G}.$$

3.3. **Partial normalisations as local contributions to the BPS invariants of stable pairs.** Consider the formula proved in Theorem 3.17 for a curve $C_G \subset X$ of arithmetic genus G satisfying the condition of minimal intersection:

$$\sum_{G\geqslant 0}\sum_{k\geqslant 1-G} e(P_k(X,[C_G]))y^k q^G = \sum_{G\geqslant 0}\sum_{g\geqslant 0} N_{g,G}y^{1-g}(1-y)^{2g-2}q^G.$$

In the previous subsection, we established that $N_{0,G}$ corresponds (up to multiplicity) to the number of rational curves in a linear system $|C|$, where $C \subset X$ is an integral curve of arithmetic genus G on a K3 surface. In this subsection, we investigate the relationship between the BPS invariants $N_{g,G}$ and the integers $n_g(C)$ studied in Section 2, for curves $C \in |C_G|$, where $C_G \subset X$ is a curve of arithmetic genus G satisfying the condition of minimal intersection.

Theorem 3.19. *Let* $\mathrm{Hilb}^k_{\mathcal{C}_G/\Pi_G}$ *be the relative Hilbert scheme of* k*-points associated to the family* $\mathcal{C}_G \longrightarrow \Pi_G = |C_G|$*. Then, we have an isomorphism* $P_{k+1-G}(X, [C_G]) \simeq \mathrm{Hilb}^k_{\mathcal{C}_G/\Pi_G}$.

Proof. Follows by Lemma 3.15. □

The above isomorphism yields

$$\sum_{k\geqslant 1-G} e(\mathrm{Hilb}^{k+G-1}_{\mathcal{C}_G/\Pi_G})y^k = \sum_{g\geqslant 0} N_{g,G} y^{1-g}(1-y)^{2g-2}, \tag{3.13}$$

where $\mathcal{C}_G \longrightarrow \Pi_G = |C_G|$ denotes the family of curves associated to the linear system $|C_G|$.

Recall that, in Section 2 we proved the relation

$$\sum_{k\geqslant 1-G} e(C^{[k+G-1]})y^k = \sum_{0\leqslant g\leqslant G} n_g(C) y^{1-g}(1-y)^{2g-2}, \tag{3.14}$$

where C is an integral Gorenstein curve of arithmetic genus G and $n_g(C)$'s are integers counting (modulo multiplicity) partial normalisations of C.

Via the adjunction formula, any element in the linear system $|C_G|$ is Gorenstein. Consider the stratification of $\Pi_G = |C_G|$ by the topological type of the curves on it, say $\Pi_G = \bigsqcup_n T_n$. Then, we obtain

$$e(\mathrm{Hilb}^{k+G-1}_{\mathcal{C}_G/\Pi_G}) = \sum_n e(C^{[k+G-1]}, C \in T_n) e(T_n).$$

Hence, by the equation (3.14) we have

$$\sum_{k\geqslant 1-G} e(\mathrm{Hilb}^{k+G-1}_{\mathcal{C}_G/\Pi_G})y^k = \sum_{0\leqslant g\leqslant G} \sum_n n_g(C \in T_n) e(T_n) y^{1-g}(1-y)^{2g-2}.$$

Comparing the previous equation with equation (3.13), we obtain the following result.

Lemma 3.20. *Let* $C_G \subset X$ *be a curve of arithmetic genus* G *satisfying the condition of minimal intersection. The integers* $N_{g,G}$ *from Theorem 3.17 satisfy following relation for* $0 \leqslant g \leqslant G$

$$N_{g,G} = \sum_n n_g(C \in T_n) e(T_n),$$

where $n_g(C \in T_n)$ *correspond to the integers from Theorem 2.6, where* $C \in T_n \subset |C_G|$*. Furthermore, as* $n_g(C) = 0$ *for* $g > G$*, we have*

$$N_{g,G} = 0$$

for $g > G$.

□

4. BPS and Gromov–Witten invariants

In this section, we explore the relation between the BPS invariants $N_{g,G}$ from Section 3.1 and the Gromov-Witten invariants via the MNOP conjecture for Calabi–Yau 3-folds.

Let Y be a Calabi–Yau 3-fold, $\beta \in H_2(Y,\mathbb{Z})$ be a non-zero curve class and let $P_k(Y,\beta)$ be the moduli space of stable pairs with Euler characteristic k and curve class β. Set

$$Z_{P,\beta}(q) = \sum_k (-1)^{\dim(P_k(X,\beta))} e(P_k(Y,\beta))q^k.$$

Additionally, let $G_{h,\beta}(Y)$ be the genus h disconnected Gromov–Witten invariant with no contracted contributions. For a summary on Gromov–Witten theory, see [PT14]. Consider the generating function

$$Z_{GW,\beta}(u) = \sum_h G_{h,\beta}(Y)u^{2h-2}.$$

Conjecture 4.1 (MNOP conjecture)**.** *In the setting described above, we have*

$$Z_{P,\beta}(-e^{iu}) = Z_{GW,\beta}(u).$$

Let X be a K3 surface and consider the Calabi–Yau 3-fold $X \times \mathbb{C}$. Let us use the MNOP conjecture to express the Gromov–Witten invariants of the Calabi–Yau $Y = X \times \mathbb{C}$, $G_{h,\beta}(X \times \mathbb{C})$, in terms of the BPS invariants for K3 surfaces explored in the previous sections.

Theorem 4.2. *Let $[C_G]$ be a curve class on X with minimal positive intersection with the polarisation H of X. Let $P_k(X \times \mathbb{C}, [C_G])$ be the moduli space of stable pairs $(\mathcal{F}, s)$ on $X \times \mathbb{C}$ with discrete invariants $\chi(\mathcal{F}) = k$ and curve class $[\mathcal{F}] = [C_G]$, whose support is contained in a fibre of the projection $X \times \mathbb{C} \longrightarrow \mathbb{C}$ (the condition on the curve class should be understood modulo the isomorphism $X \simeq X \times \{t\}$). Then,*

$$P_k(X \times \mathbb{C}, [C_G]) \simeq P_k(X, [C_G]) \times \mathbb{C}.$$

Proof. Since C_G lies on a fibre of the projection $X \times \mathbb{C} \longrightarrow \mathbb{C}$, the elements in the curve class $[C_G]$ are Gorenstein. Then, by Theorem 3.15, stable pairs $(\mathcal{F}, s)$ can be completely specified by $\mathrm{supp}(\mathcal{F})$ and $\mathrm{supp}(\mathrm{coker}(s))$. Thus, we have the inclusion

$$P_k(X, [C_G]) \times \mathbb{C} \hookrightarrow P_k(X \times \mathbb{C}, [C_G]), \ (\mathcal{F}, s, t \in \mathbb{C}) \longmapsto (\mathcal{F}, s) \in X \times \{t\}. \tag{4.1}$$

Note that the only obstruction for the above morphism to be an isomorphism is that a pair $(\mathcal{F}, s) \in P_k(X \times \mathbb{C}, [C_G])$ might be scheme theoretically supported on a thickening of a fibre, as this would introduce an extra degree of freedom. However, this case never occurs. Indeed, since C_G satisfies the condition of minimal intersection, $\mathcal{F}$ is stable, and so $End(\mathcal{F}) \simeq \mathbb{C}$. If $\mathcal{F}$ is supported on a thickening $C \times \mathrm{Spec}(\mathbb{C}[x]/(x^n))$ (with $C \subset X$), we can define the endomorphism $\mathcal{F} \xrightarrow{\cdot x} \mathcal{F}$, which is nilpotent. This contradicts the simplicity of $\mathcal{F}$. □

By Theorem 3.2, $P_k(X,[C_G])$ is smooth of dimension $3-2G+k$, so $\dim(P_k(X\times\mathbb{C},[C_G]))+k$ is even. Then, for $\beta=[C_G]$ with C_G contained in a fibre of the projection $X\times\mathbb{C}\longrightarrow\mathbb{C}$, the previous Theorem yields

$$Z_{P,\beta}(-e^{iu})=\sum_k e(P_k(X,\beta))e^{iku}.$$

By Theorem 3.17 and Lemma 3.20, we have

$$\sum_k e(P_k(X,\beta))y^k=\sum_{0\leqslant g\leqslant G} N_{g,G}(X)y^{1-g}(1-y)^{2g-2}.$$

Thus, assuming the MNOP conjecture, we obtain the relation

$$\sum_{0\leqslant g\leqslant G} N_{g,G}(X)(-1)^{g-1}2^{2g-2}\cos(u/2)^{2g-2}=\sum_{h\geqslant 0} G_{h,\beta}(X\times\mathbb{C})u^{2h-2},$$

which allows us to express the Gromov-Witten invariants of the Calabi–Yau 3-fold $X\times\mathbb{C}$ in terms of the BPS invariants of the K3 surface X. For example, for $u=0$ we obtain

$$G_{1,\beta}(X\times\mathbb{C})=\sum_{0\leqslant g\leqslant G} N_{g,G}(X)(-1)^{g-1}2^{2g-2}. \tag{4.2}$$

Note that the invariants $N_{g,G}(X)$ depend only on β^2. Hence, equation (4.2) shows that the invariants $G_{1,\beta}(X\times\mathbb{C})$ also depend only on β^2. Moreover, these relation also imply that the Gromov–Witten invariants $G_{1,\beta}(X\times\mathbb{C})$ are rational numbers and, in general, not integers.

Appendix 1: On the moduli space of (semi)stable sheaves

In this appendix, we recall some basic facts about the moduli space of (semi)stable sheaves on K3 surfaces following [HL10]. In particular, we outline the construction of the symplectic structure on the moduli space of stable sheaves.

Let X be a K3 surface. A sheaf $\mathcal{E}\in\mathrm{Coh}(X)$ is said to be *pure of dimension d* if for all non-trivial coherent subsheaves $\mathcal{F}\subset\mathcal{E}$, we have $\dim(\mathcal{F})=d$. Equivalently, $\mathcal{E}$ is pure if all its associated points have the same dimension. The following characterisation has played an important role in this thesis.

Proposition 5.1. *Let $\mathcal{E}\in\mathrm{Coh}(X)$ of codimension c. Then, $\mathcal{E}$ is pure if, and only if*

$$\mathrm{codim}(\mathcal{E}xt^q(\mathcal{E},\omega_X))\geqslant q+1$$

for all $q>c$.

Let H be a fixed polarisation of X. With respect to H, we consider two notions of stability: Gieseker's and slope stability.

Definition 5.2 (Gieseker stability)**.** For $\mathcal{E}\in\mathrm{Coh}(X)$ of dimension d, let $P(\mathcal{E})$ be its Hilbert polynomial with coefficients α_i, and denote by $p(\mathcal{E})=P(\mathcal{E})/\alpha_d$ its reduced Hilbert polynomial. We say that $\mathcal{E}$ is (semi)stable if it is pure and for all proper subsheaves $\mathcal{F}\subset\mathcal{E}$ we have

$$p(\mathcal{F})(\leqslant)<p(\mathcal{E}),$$

where for polynomials $f(m),g(m)$ we write $f\leqslant g$ if $f(m)\leqslant g(m)$ for $m\gg 0$.

Definition 5.3. Let $\mathcal{E} \in \mathrm{Coh}(X)$ of dimension 2. We define the slope of $\mathcal{E}$ as

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H}{\mathrm{rk}(\mathcal{E})},$$

where $c_1(\mathcal{E})$ denotes the first Chern class of $\mathcal{E}$ and $\mathrm{rk}(\mathcal{E})$ its rank. We say that $\mathcal{E}$ is μ-(semi)stable if any torsion subsheaf of $\mathcal{E}$ has codimension 2 and

$$\mu(\mathcal{F})(\leqslant) < \mu(\mathcal{E}),$$

for all subsheaves $\mathcal{F} \subset \mathcal{E}$ of positive rank. In general, given $\mathcal{E} \in \mathrm{Coh}(X)$ of dimension d, we define $\hat{\mu}(\mathcal{E}) := \frac{\alpha_{d-1}}{\alpha_d}$ [3], where α_i's denote the coefficients of the Hilbert polynomial of $\mathcal{E}$.

The two stability conditions are related in the following form.

Lemma 5.4. *If $\mathcal{E}$ is a pure coherent sheaf of dimension 2, then we have the following chain of implications*

$$\mathcal{E} \textit{ is } \mu\textit{-stable} \implies \mathcal{E} \textit{ is stable} \implies \mathcal{E} \textit{ is semistable} \implies \mathcal{E} \textit{ is } \mu\textit{-semistable.}$$

Furthermore, if $\mathcal{E}$ is μ-semistable with $c_1(\mathcal{E}) \cdot H$ and $\mathrm{rk}(\mathcal{E})$ coprime, $\mathcal{E}$ is μ-stable.

It is an important fact that stable sheaves are simple.

Proposition 5.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be semistable coherent sheaves of the same dimension. If $p(\mathcal{F}) > p(\mathcal{G})$, then $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$. If $\mathcal{F}$ is stable and $p(\mathcal{F}) = p(\mathcal{G})$, a non-trivial morphism $f \colon \mathcal{F} \longrightarrow \mathcal{G}$ is is injective. If $\mathcal{G}$ is stable and $p(\mathcal{F}) = p(\mathcal{G})$, a non-trivial morphism $f \colon \mathcal{F} \longrightarrow \mathcal{G}$ is surjective. Furthermore, if $\mathcal{F}, \mathcal{G}$ are stable with $P(\mathcal{F}) = P(\mathcal{G})$, any non-trivial morphism $f \colon \mathcal{F} \longrightarrow \mathcal{G}$ is an isomorphism.*

Corollary 5.6. *If $\mathcal{E}$ is a stable sheaf, $\mathrm{End}(\mathcal{E})$ is a finite dimensional division algebra over $\mathbb{C}$. Hence, $\mathrm{End}(\mathcal{E}) \simeq \mathbb{C}$.*

The moduli space of (semi)stable sheaves. We are interested in parametrising (semi)stable sheaves with fixed numerical invariants specified by a so called Mukai vector on a polarised K3 surface (X, H).

Definition 5.7. Consider the lattice

$$H^{\mathrm{ev}}(X, \mathbb{Z}) := H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$

with the pairing given by

$$\langle v, w \rangle := v_0 w_2 - v_1 w_1 + v_2 w_0,$$

where $v = (v_0, v_1, v_2)$, $w = (w_0, w_1, w_2) \in H^{\mathrm{ev}}(X, \mathbb{Z})$. This lattice is called *Mukai lattice*.
The *Mukai vector* of $\mathcal{E} \in \mathrm{Coh}(X)$ is defined by

$$v(\mathcal{E}) := ch(\mathcal{E})\sqrt{Td(X)} = (r, c_1, \frac{c_1^2}{2} - c_2 + r),$$

where $ch(\mathcal{E})$ denotes the Chern character of $\mathcal{E}$, r is its rank, c_1 is its first Chern class and c_2 is its second Chern class. We say that the Mukai vector v is *primitive* if it is not divisible by any integer $m > 1$.

[3] Note that $\mu(\mathcal{E}) = \alpha_d(\mathcal{O}_X)\hat{\mu}(\mathcal{E}) - \alpha_{d-1}(\mathcal{O}_X)$, so the two slopes do not coincide in general.

Consider the following moduli functor

$$\mathcal{M}(v)\colon (\mathrm{Sch}/\mathbb{C})^{op} \longrightarrow \mathrm{Sets},$$

where for $S \in \mathrm{Sch}/\mathbb{C}$, we set $\mathcal{M}(v)(S)$ to be the set of isomorphism classes of S-flat families of semistable sheaves on X with Mukai vector v modulo the relation $\sim$. We say $\mathcal{F} \sim \mathcal{G}$ if there exists $\mathcal{L} \in \mathrm{Pic}(S)$ with $\mathcal{F} \otimes p^*\mathcal{L} \simeq \mathcal{G}$. Additionally, for $f\colon S' \longrightarrow S$ let $\mathcal{M}(f)$ be be the pull-back along $f \times \mathrm{id}_X$. Similarly, we define the open subfunctor of stable sheaves $\mathcal{M}^s(v) \subset \mathcal{M}(v)$.
A scheme $M(v)$ correpresenting the moduli functor $\mathcal{M}(v)$ is called *moduli space of semistable sheaves of Mukai vector v*. Gieseker constructed the moduli space $\mathcal{M}(v)$ of stable sheaves on X as a projective scheme, see [Gie77].

Theorem 5.8. *The moduli space of stable sheaves $\mathcal{M}^s(v)$ is smooth of dimension $\langle v, v \rangle + 2$.*

Proof. The obstruction of deforming $\mathcal{E} \in \mathcal{M}^s(v)$ lies in $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$, see [HL10]. Consider the trace map

$$tr\colon \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \longrightarrow H^2(X,\mathcal{O}_X),$$

which is an isomorphism for simple sheaves, as it is the dual of $H^0(X,\mathcal{O}_X) \simeq \mathbb{C} \longrightarrow \mathrm{Hom}(\mathcal{E},\mathcal{E})$. Via this map, the obstruction to deforming $\mathcal{E} \in \mathcal{M}^s(v)$ gives an obstruction to deforming the line bundle $\det(\mathcal{E})$. On a K3 surface, the obstruction to deforming line bundles vanishes. Hence, the obstruction to deform $\mathcal{E}$ also vanishes.
Furthermore, the Zariski tangent space at $\mathcal{E} \in \mathcal{M}^s(v)$ is identified with $\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$, see [HL10]. By Grothendieck-Riemann-Roch, we obtain

$$-\langle v, v \rangle = \chi(\mathcal{E},\mathcal{E}) = \dim \mathrm{Hom}(\mathcal{E},\mathcal{E}) - \dim \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) + \dim \mathrm{Ext}^2(\mathcal{E},\mathcal{E}).$$

Via Serre duality, $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \simeq \mathrm{Hom}(\mathcal{E},\mathcal{E})^\vee \simeq \mathbb{C}$. Hence, the dimension of the moduli space of stable sheaves is $\mathcal{M}^s(v)$ is $\langle v, v \rangle + 2$. □

Given a Mukai vector v, we say that a polarisation H is *v-general* if it does not lie in any wall in the ample cone of X. Details can be found in Section 4.C. in [HL10].

Proposition 5.9. *Let $\mathcal{E} \in \mathrm{Coh}(X)$ be a semistable sheaf. If its Mukai vector $v(\mathcal{E})$ is primitive and the polarisation H is v-general, $\mathcal{E}$ is stable.*

Symplectic structure on the moduli of stable sheaves. We present here an sketch of the construction of the symplectic structure on the moduli space of stable sheaves, and refer to [HL10] for details.

Stable sheaves are simple and the tangent space of the moduli space $\mathcal{M}(v)$ at a simple sheaf is given by $T_{\mathcal{E}}\mathcal{M}(v) \simeq \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$. Given $\alpha \in H^0(X,\omega_X)$, we define

$$\tau(\alpha)\colon \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \xrightarrow{\circ} \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{tr} H^2(X,\mathcal{O}_X) \xrightarrow{\alpha} H^0(X,\omega_X) \simeq \mathbb{C},$$

where $\circ$ denotes the Yoneda cup product and tr the trace. For holomorphicity, see [Muk88] pp. 154. The question whether the two form $\tau(\alpha)$ is non-degenerate is reduced to the following local result, see [HL10].

Proposition 5.10 ([HL10]). *Let $\mathcal{E} \in M^s(v)$. The 2-form $\tau(\alpha)(\mathcal{E})$ is non-degenerate if, and only if multiplication by α induces an isomorphism $\alpha_*\colon \operatorname{Ext}^1_X(\mathcal{E},\mathcal{E}) \longrightarrow \operatorname{Ext}^1_X(\mathcal{E},\mathcal{E}\otimes K_X)$. Hence, by picking α as $\mathcal{O}_X \xrightarrow{1} \mathcal{O}_X \simeq K_X$, we obtain the non-degeneracy of the symplectic structure.*

Proof. Let $\mathcal{E}^\bullet \longrightarrow \mathcal{E}$ be a finite locally free resolution of $\mathcal{E}$ and let $\mathcal{A}^\bullet = \mathcal{H}om^\bullet(\mathcal{E}^\bullet,\mathcal{E}^\bullet)$, where we define $\mathcal{H}om^i(\mathcal{E}^\bullet,\mathcal{E}^\bullet) = \oplus_k \mathcal{H}om(\mathcal{E}^k,\mathcal{E}^{k+i})$ with boundary operators $d\varphi = d_{\mathcal{E}}\circ\varphi - (-1)^{deg\varphi}\varphi\circ d_{\mathcal{E}}$. Then,

$$\mathcal{A}^\bullet \otimes \mathcal{A}^\bullet \xrightarrow{\circ} \mathcal{A}^\bullet \xrightarrow{tr_{\mathcal{E}^\bullet}} \mathcal{O}_X$$

is a perfect pairing and leads to an isomorphism $\mathcal{A}^\bullet \longrightarrow \mathcal{H}om^\bullet(\mathcal{A}^\bullet,\mathcal{O}_X)$. Furthermore, for a section $\alpha\colon \mathcal{O}_X \longrightarrow K_X$, we have the commutative diagram

$$\begin{array}{ccccc}
(\mathcal{A}^\bullet \otimes K_X)\otimes\mathcal{A}^\bullet & \xrightarrow{\simeq} & \mathcal{H}om^\bullet(\mathcal{A}^\bullet,K_X)\otimes\mathcal{A}^\bullet & \xrightarrow{ev} & K_X \\
{\scriptstyle (1\otimes\alpha)\otimes 1}\uparrow & & & & \uparrow{\scriptstyle \alpha} \\
\mathcal{A}^\bullet\otimes\mathcal{A}^\bullet & \xrightarrow{\quad\circ\quad} & \mathcal{A}^\bullet & \xrightarrow{\quad tr \quad} & \mathcal{O}_X ,
\end{array}$$

where the morphism $ev\colon \mathcal{H}om^\bullet(\mathcal{A}^\bullet,K_X)\otimes\mathcal{A}^\bullet \longrightarrow K_X$ is given by $\phi\otimes a \longmapsto \phi(a)$. This morphisms of complexes induce morphisms in cohomology that make the following diagram commute

$$\begin{array}{ccccc}
\operatorname{Ext}^i_X(\mathcal{E},\mathcal{E}\otimes K_X)\otimes\operatorname{Ext}^j_X(\mathcal{E},\mathcal{E}) & \xrightarrow{\simeq} & \mathbb{E}xt^i_X(\mathcal{A}^\bullet,K_X)\otimes\mathbb{H}^j(\mathcal{A}^\bullet) & \longrightarrow & H^{i+j}(X,K_X) \\
{\scriptstyle \alpha_*\otimes 1}\uparrow & & & & \uparrow{\scriptstyle \alpha} \\
\operatorname{Ext}^i_X(\mathcal{E},\mathcal{E})\otimes\operatorname{Ext}^j_X(\mathcal{E},\mathcal{E}) & \xrightarrow{\quad\circ\quad} & \operatorname{Ext}^{i+j}_X(\mathcal{E},\mathcal{E}) & \xrightarrow{\quad tr\quad} & H^{i+j}(X,\mathcal{O}_X) .
\end{array}$$

Note that for $i = j = 1$, $\tau(\alpha)(\mathcal{E})$ is the map from the lower left corner of the diagram to the upper right corner.
Since X is a smooth surface and $\mathcal{A}^\bullet$ is a bounded complex of coherent sheaves, Serre duality ensures that the pairing

$$\mathbb{E}xt^{2-1}(\mathcal{A}^\bullet,K_X)\otimes\mathbb{H}^1(X,\mathcal{A}^\bullet) \longrightarrow H^2(X,K_X) \xrightarrow{\simeq} K$$

is a perfect pairing. Hence, following the commutative diagram for the case $i = j = 1$, we conclude that $\tau(\alpha)$ is non-degenerate if, and only if α_* is an isomorphism. □

For closedness of the 2-form see Proposition 10.3.2. in [HL10].

Appendix 2: Euler characteristic of the compactified Jacobian of a curve

In this appendix, we study the topological Euler characteristic of the compactified Jacobian of a rational curve C, and present explicit calculations for the case of C having only simple singularities. This appendix follows closely [Bea97].

In Proposition 1.9, we saw that given an integral rational curve C and its minimal unibranch partial normalisation $\hat{C} \longrightarrow C$, we have

$$e(\overline{\operatorname{Pic}^0}(C)) = e(\overline{\operatorname{Pic}^0}(\hat{C})).$$

Hence, we may assume that C is rational unibranch.

Let $x \in C$, $\delta_x := \dim_{\mathbb{C}}(\mathcal{O}_{\tilde{C},x}/\mathcal{O}_{C,x})$ and $c := \mathcal{O}_{\tilde{C}}(\sum_x (2\delta_x)[x])$, where $\tilde{C}$ denotes the normalisation of C. Define the finite dimensional algebras $A_x := \mathcal{O}_{C,x}/c_x$, $\tilde{A}_x := \mathcal{O}_{\tilde{C},x}/c_x$, and let $\mathbb{G}(\delta_x, \tilde{A}_x)$ be the Grassmannian of codimension δ_x subspaces of $\tilde{A}_x$. Furthermore, let $\mathbb{G}_x \subset \mathbb{G}(\delta_x, \tilde{A}_x)$ be the closed subvariety consisting of the elements that are A_x-modules. Since any $\mathcal{O}_{C,x}$-module contains c_x, see Lemma 1 in [GPL97], $\mathbb{G}_x$ parameterises $\mathcal{O}_{C,x}$-modules $\mathcal{L}_x$, which have codimension δ_x as submodules of $\mathcal{O}_{\tilde{C},x}$.

Note that $\mathcal{O}_{\tilde{C}}/c$ is a skyscraper sheaf with fibre $\tilde{A}_x$ at x, hence $\prod_{x\in\Sigma} \mathbb{G}_x$ parametrises $\mathcal{O}_C$-modules $\mathcal{L}$, which are submodules of $\mathcal{O}_{\tilde{C}}$ with $\dim(\mathcal{O}_{\tilde{C},x}/\mathcal{L}_x) = \delta_x$ for all $x \in C$. Thus, given $\mathcal{L} \in \prod_{x\in\Sigma} \mathbb{G}_x$, we have that $\chi(\mathcal{O}_{\tilde{C}}/\mathcal{L}) = \sum_x \delta_x = \chi(\mathcal{O}_{\tilde{C}}/\mathcal{O}_C)$, which implies that $\mathcal{L} \in \overline{\mathrm{Pic}^0}(C)$. So, we have a morphism

$$e\colon \prod_{x\in\Sigma} \mathbb{G}_x \longrightarrow \overline{\mathrm{Pic}^0}(C).$$

Proposition 6.1 ([Bea97]). *The morphism* $e\colon \prod_{x\in\Sigma} \mathbb{G}_x \longrightarrow \overline{\mathrm{Pic}^0}(C)$ *constructed above is a homeomorphism.*

Proof. Via the adjunction formula we have

$$p_a(\mathrm{Bl}_x C) = p_a(C) - \frac{r(r-1)}{2},$$

where $\mathrm{Bl}_x C$ denotes the blow-up of C at x and r is the multiplicity of the singularity $x \in C$. Hence, since $p_a(\tilde{C}) \geqslant 0$, by applying the previous equation successively, we get a bound on the number of singularities, as well as on their multiplicities. Hence, $\prod_{x\in\Sigma} \mathbb{G}_x$ is compact. Since both varieties are compact, it is enough to prove that e is a bijection.
For the injectivity, let $\mathcal{L}, \mathcal{M} \in \prod_{x\in\Sigma} \mathbb{G}_x$ with $e(\mathcal{L}) = e(\mathcal{M})$. Then, $\mathcal{L} \simeq \mathcal{M}$, which implies that there exists a rational section $s \in \mathcal{O}_{\tilde{C}}$ such that $\mathcal{M} = s\mathcal{L}$. By definition of $\mathbb{G}_x$, we also have that $\dim(\mathcal{O}_{\tilde{C},x}/\mathcal{L}_x) = \delta_x = \dim(\mathcal{O}_{\tilde{C},x}/\mathcal{M}_x)$. Hence,

$$dim(\mathcal{O}_{\tilde{C},x}/\mathcal{M}_x) = dim(\mathcal{O}_{\tilde{C},x}/\mathcal{L}_x) = \dim(s_x\mathcal{O}_{\tilde{C},x}/\mathcal{M}_x),$$

which implies that $\dim(\mathcal{O}_{\tilde{C},x}/s_x\mathcal{O}_{\tilde{C},x})$ is zero, and hence $\mathcal{O}_{\tilde{C},x} = s_x\mathcal{O}_{\tilde{C},x}$ for all x, and so s must be constant.
For the surjectivity, let $\pi\colon \tilde{C} \longrightarrow C$ be the normalisation and let $\mathcal{L} \in \overline{\mathrm{Pic}^0}(C)$. Denote by $\tilde{\mathcal{L}} = \pi^*\mathcal{L}/T(\pi^*\mathcal{L})$. We claim that $\deg(\tilde{\mathcal{L}}) \leqslant 0$. Consider the exact sequence [4]

$$0 \longrightarrow \mathcal{L} \longrightarrow f_*\tilde{\mathcal{L}} \longrightarrow \tau \longrightarrow 0$$

where τ a skyscraper sheaf supported on the singular locus of C, such that $\tau_x \leqslant \delta_x$ for all $x \in C$. Thus, we have $\chi(\tilde{\mathcal{L}}) - \chi(\mathcal{L}) \leqslant \chi(\mathcal{O}_{\tilde{C}}) - \chi(\mathcal{O}_C)$, which implies

$$\deg(\tilde{\mathcal{L}}) = \chi(\tilde{\mathcal{L}}) - \chi(\mathcal{O}_{\tilde{C}}) \leqslant \chi(\mathcal{L}) - \chi(\mathcal{O}_C) = 0.$$

[4] For the proof of exactness, see Lemma 1 in [GPL97]

Now, since $\tilde{C}$ is rational, $\tilde{\mathcal{L}}^{-1} = \mathcal{O}(-\deg(\mathcal{L}))$. Thus, it has a global section whose zero locus is contained in the singular locus of C. Hence, using the isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{L}, \mathcal{O}_{\tilde{\mathrm{C}}}) \simeq \mathrm{Hom}_{\mathcal{O}_{\tilde{\mathrm{C}}}}(\pi^*\mathcal{L}, \mathcal{O}_{\tilde{\mathrm{C}}}) \simeq \mathrm{Hom}_{\mathcal{O}_{\tilde{\mathrm{C}}}}(\tilde{\mathcal{L}}, \mathcal{O}_{\tilde{\mathrm{C}}}),$$

we conclude that there exists a morphism $i\colon \tilde{\mathcal{L}} \longrightarrow \mathcal{O}_{\tilde{\mathrm{C}}}$ which is bijective outside of Σ because the associated section has only zeros on Σ. Let $n_x := \dim(\mathcal{O}_{\tilde{\mathrm{C}},x}/i(\mathcal{L}_x))$ for each $x \in \Sigma$. Then, we have $\sum_{x\in\Sigma} n_x = \dim(\mathcal{O}_{\tilde{\mathrm{C}}}/i(\mathcal{L}))$ as $\mathcal{O}_{\tilde{\mathrm{C}}}/i(\mathcal{L})$ is supported on Σ with fibres of dimension n_x. Then, since $\sum_x \delta_x = \chi(\mathcal{O}_{\tilde{\mathrm{C}}}) - \chi(\mathcal{O}_C) = g$, we have

$$\sum_{x\in\Sigma} n_x = \dim(\mathcal{O}_{\tilde{\mathrm{C}}}/i(\mathcal{L})) = \chi(\mathcal{O}_{\tilde{\mathrm{C}}}) - \chi(\mathcal{L}) = g = \sum_{x\in\Sigma} \delta_x.$$

□

The variety $\mathbb{G}_x$ depends only on the completion of the local ring $\mathcal{O}_{C,x}$. We have seen that $e(\mathbb{G}_x)$ parametrises the sub-$\mathcal{O}_{C,x}$-modules L of the normalisation $\tilde{\mathcal{O}_{C,x}}$ of $\mathcal{O}_{C,x}$ satisfying $\dim(\tilde{\mathcal{O}_{C,x}}/L) = \dim(\tilde{\mathcal{O}_{C,x}}/\mathcal{O}_{C,x})$.

Corollary 6.2. *Let C be a rational unibranch curve. Then, $e(\overline{\mathrm{Pic}^0}(C)) = \prod_{x\in\Sigma} \epsilon(x)$, where $\epsilon(x) := e(\mathbb{G}_x)$ and Σ denotes the singular locus of C.*

Proof. The previous proposition guarantees that $e(\overline{\mathrm{Pic}^0}(C)) = \prod_{x\in C} \epsilon(x)$. Furthermore, note that if x is a smooth point, then $\mathbb{G}_x = \{e\}$ because by definition $\mathbb{G}_x$ parametrises sub-$\mathcal{O}_{C,x}$-modules $\mathcal{L}$ of the normalization $\mathcal{O}_{\tilde{C},x}$ of $\mathcal{O}_{C,x}$ with $\dim(\mathcal{O}_{\tilde{\mathrm{C}},x}/\mathcal{L}_x) = \dim(\mathcal{O}_{\tilde{C},x}/\mathcal{O}_{C,x})$. Hence, $\epsilon(x) = 1$ if x is a smooth point of C, and the product runs over the singular locus $\Sigma \subset C$. □

Proposition 6.3. *Let p, q be two coprime integers. If the singularity $x \in C$ has as local model $\mathbb{C}[[x, y]]/(x^p - y^q)$, we obtain*

$$\epsilon(x) = \frac{1}{p+q}\binom{p+q}{p}.$$

Proposition 6.4 ([Bea97])**.** *Let C be a rational curve and let $x \in C$ be a simple singularity. Then, $\epsilon(x)$ is the number of isomorphism classes of torsion free rank 1 $\mathcal{O}_{C,x}$-modules., and we have:*

- $\epsilon(x) = l + 1$ *if x is of type A_{2l};*
- $\epsilon(x) = 1$ *if x is of type A_{2l+1};*
- $\epsilon(x) = 1$ *if x is of type D_{2l} ($l \geqslant 2$);*
- $\epsilon(x) = l$ *if x is of type D_{2l+1} ($l \geqslant 2$);*
- $\epsilon(x) = 5$ *if x is of type E_6;*
- $\epsilon(x) = 2$ *if x is of type E_7;*
- $\epsilon(x) = 7$ *if x is of type E_8.*

Proof. Assume that C has only one singularity with local ring $\mathcal{O}_{C,x}$. Consider the natural action of $\mathrm{Pic}^0(C)$ on $\overline{\mathrm{Pic}^0}(C)$, which has finitely many orbits corresponding to the different isomorphism classes of rank 1 $\mathcal{O}_{C,x}$-modules. Since the orbits of the action are of the form $\mathbb{A}^n$, they have Euler characteristic 1, and so $\epsilon(x) = \epsilon(\overline{\mathrm{Pic}^0}(C))$ equals the number of these orbits. Since $\mathcal{O}_{C,x}$ is unibranch, its completion is of the form $\mathbb{C}[[x, y]]/(x^p - y^q)$ with $p = 2$, $q = 2l + 1$

for the type A_{2l}, $p=3, q=4$ for the type E_6, and $p=3$, $q=5$ for the type E_8. Then, the claimed result follows from Proposition 6.3.
If the singularity of C is of type A_{2l+1}, it has local model defined by $x^2-y^{2l}=0$. Locally around such a singularity, the curve C is the union of two smooth branches with a high order contact, so by Proposition 1.9 we have $\epsilon(x)=1$. A D_l singularity is the union of a A_{l-3} branch and a transversal smooth branch, hence we have the result by Proposition 1.9. Finally an E_7 singularity is the union of an ordinary cusp and its tangent, hence it has $\epsilon(x)=2$. □

Appendix 3: Proof of Theorem 3.10

Yoshioka proved that if $r>0$, or $r=0$ and C_G ample with $C_G^2=2G-2$, the moduli space of μ-semistable sheaves on a K3 surface X, $\mathcal{M}(r,[C_G],a))$, and the Hilbert scheme of points $X^{[G-ra]}$ are deformation equivalent, see Theorem 0.2. in [Yos99]. In this appendix, we present a modification of Yoshioka's proof of Theorem 0.2. to include the case $r=0$ for C_G satisfying the condition of minimal intersection.

Theorem 7.1 ([KY00]). *Let $C_G\subset X$ satisfying the condition of minimal intersection. Then, the moduli space of μ-stable sheaves $\mathcal{M}(0,[C_G],a)$ is deformation equivalent to the Hilbert scheme of points $X^{[G]}$. In particular, we have*

$$\chi_{tt'}(\mathcal{M}(0,[C_G],a))=\chi_{tt'}(X^{[G]}),$$

where $\chi_{tt'}(-)$ denotes the Hodge polynomial.

We first consider the following intermediate results.

Theorem 7.2 ([Yos99]). *Let v, v_1 be Mukai vectors with $\langle v_1^2\rangle=-2$. Then, for $w=-R_{v_1}(v)$ we have the isomorphism $\mathcal{M}_H(v)\simeq\mathcal{M}_H(w)$, if v_1, v satisfy:*
(1) $-[v_1]_0\langle v_1,v\rangle-[v]_0>0$,
(2) $[v]_0>\langle v,v\rangle/2+1$, and
(3) $-\langle v_1,v\rangle>\langle v,v\rangle/2$.

Theorem 7.3 ([Yos99]). *Let X_1, X_2 be two K3 surfaces and let $v_1=(lr,l\eta_1,a_1)\in H^*(X_1,\mathbb{Z})$ and $v_2=(lr,l\eta_2,a_2)\in H^*(X_2,\mathbb{Z})$ be primitive Mukai vectors such that:*
(1) $r,l>0$,
(2) $r+\eta_1$ and $r+\eta_2$ are primitive, and
(3) $\langle v_1,v_1\rangle=\langle v_2,v_2\rangle$.
(4) $a_1=a_2 \mod l$
Then, $\mathcal{M}_{H_1}(v_1)$ and $\mathcal{M}_{H_2}(v_2)$ are deformation equivalent.

Theorem 7.4 ([Yos99]). *Let $v = (r, \eta, a)$ Mukai vector such that (r, η) is primitive with $r > 0$ and $\langle v, v \rangle \geqslant -2$. Then, for a general H, $\mathcal{M}_H(v) \neq \varnothing$ and $\mathcal{M}_H(v)$ is deformation equivalent to $Hilb^{\langle v,v \rangle/2+1}$.*

Proof. By Theorem 7.3 and the fact that the Hilbert schemes of n-points of two K3 surfaces are deformation equivalent, it is enough to assume that $\mathrm{Pic}(X) = H\mathbb{Z}$ with $H^2 = 2(ar + s)$ and $a > s + 1$ for $\langle v, v \rangle = 2s$.
Let $u = (a, -H, r)$ and let $v_1 = v(\mathcal{O}_Y)$. For $\mathcal{F} \in \mathcal{M}_H(u)$, we have $\langle v_1, u \rangle = -\chi(\mathcal{F}) = -(r + a)$. Since u satisfies the conditions of Theorem 7.2, we have

$$\mathcal{M}(u) \simeq \mathcal{M}(-R_{v_1}(u)) \text{ for } -R_{v_1}(u) = -u - \langle v_1, u \rangle v_1 = v.$$

Let now Z be a K3 surface with $\mathrm{Pic}(Z) = H'\mathbb{Z}$ and $H'^2 = 2(a + s)$. Theorem 7.3 yields $\mathcal{M}_H(u) \simeq \mathcal{M}_{H'}(a, -H', 1)$. By Theorem 7.2 applied to $v_1 = v(\mathcal{O}_X)$, we obtain an isomorphism $\mathcal{M}_{H'}(a, -H', 1) \simeq \mathcal{M}_{H'}(1, H', a)$. Thus, we have

$$\mathcal{M}_H(v) \simeq \mathcal{M}_{H'}(1, H', a).$$

Finally, we note that $\mathcal{M}_{H'}(1, H', a) \simeq \mathrm{Hilb}^{s+1}(X)$. Indeed, for $\mathcal{F} \in \mathcal{M}_{H'}(1, H', a)$ we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \det(\mathcal{F}) = \mathcal{O}_Z(H') \longrightarrow Q \longrightarrow 0.$$

Define the morphism $\mathcal{F} \longmapsto (\mathcal{O}_Z \longrightarrow Q(-H'))$. We construct the inverse as follows. Via the above exact sequence, we identify $\mathcal{F}(-H')$ with the ideal sheaf of a closed subscheme $W \subset Z$. Then, we define the inverse

$$W \in \mathrm{Hilb}^{s+1}(X) \longmapsto \mathcal{I}_W(H') \in \mathcal{M}_{H'}(1, H', a).$$

Note that $\mathcal{I}_W(H')$ is stable as it is a torsion free sheaf of rank 1 on an integral variety. $\square$

Proof of Theorem 7.1. We first reduce to the case $r > 0$. Let H be the ample line bundle from the condition of minimal intersection for C_G. After replacing $\mathcal{F} \in \mathcal{M}(0, [C_G], a)$ by $\mathcal{F} \otimes H^{\otimes n} \in \mathcal{M}(0, [C_[G]], a + n \deg(C_G))$ for n big enough, we may assume that the evaluation map $\phi \colon H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \longrightarrow \mathcal{F}$ is surjective for all $\mathcal{F} \in \mathcal{M}(0, [C_G], a)$. By Lemma 2.1 in [Yos99], $\ker(\phi)$ is μ-stable. Then, we consider the Mukai reflection with respect to $v(\mathcal{O}_X)$ given by

$$\Phi \colon \mathcal{M}(0, [C_G], a) \longrightarrow \mathcal{M}(a, -[C_G], 0),\ \mathcal{F} \longmapsto \ker(\phi).$$

The morphism Φ is a proper monomorphism, so it is a closed immersion. Since $\mathcal{M}(a, -[C_G], 0)$ is irreducible and both moduli spaces have the same dimension, Φ is an isomorphism. Finally, we conclude by Theorem 7.4. $\square$

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